NUMERICAL FUNCTIONS AND TRIPLETS

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We consider the functions: f_s, f_d, f_p, $F : \mathbf{N}^* \to \mathbf{N}$, where $f_s(k) = n$, $f_d(k) = n$, $f_p(k) = n$, $F(k) = n$, n being, respectively, the least natural number such that $k/n! - 1$, $k/n! + 1$, $k/n! \pm 1$, $k/n!$ or $k/n! \pm 1$. This functions have the next properties:

1. Obviouvsly, from definition of this function, it results:

$$F(k) = \min\{S(k), f_p(k)\} = \min\{S(k), f_s(k), f_d(k)\}$$

where S is the Smarandache function (see [3]).

2. $F(k) \leq S(k)$, $F(k) \leq f_s(k)$, $F(k) \leq f_d(k)$, $F(k) \leq f_p(k)$

3. $F(k) = S(k)$ if k is even, $k \geq 4$.
Proof. For any $n \in \mathbf{N}$, $n \geq 2$, $n!$ is even, $n! \pm 1$ are odd. If k is even, then k cannot divide $n! \pm 1$. So $F(k) = S(k) = n \geq 2$ if k is even, $k \geq 4$.

4. If $p > 3$ is prime number, then $F(p) \leq p - 2$.
Proof. According to Wilson's theorem $(p-1)! + 1 = M_p$. Because $(p-2)! - 1 + (p-1)! + 1 = (p-2)!p$ results for $p > 3$, $(p-2)! - 1 = M_p$ and so $F(p) \leq p - 2$.

5. $F(m!) = F(m! \pm 1) = S(m!) = m$.

6. The equation $F(k) = F(k+1)$ has infinitly many solutions, because, according to the property 5), there is the solutions $k = m!$, $m \in \mathbf{N}^*$.

7. If $F(k) = S(k)$ and n is the least natural number such that $k/n!$, then k not divide $s! \pm 1$ for $s < n$.
Let $k = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. According to $S(k) = \max\limits_{1 \le i \le r}\{S_{p_i}(\alpha_i)\}$, it results that $S(k) \ge p_h$, where $p_h = \min\{p_1, p_2, \dots, p_r\}$.
If k not divide $s! \pm 1$ for $s \le p_h$, then k not divide $t! \pm 1$ for $t > p_h$.
Consequently, if k not divide $(n-1)!$, $k/n!$ and k not divide $s! \pm 1$ for $s \le \min\{n, p_h\}$, then $F(k) = S(k) = n$.
Obviously, the numbers $k = 3t$, t being odd, $t \ne 1$, have $p_h = 3$ and they satisfy the condition $3t$ not divide $s! \pm 1$ for $s = 1, 2, 3$.
Therefore, for $k = 3t$, t odd, $t \ne 1$, $F(3t) = S(3t) = n$, n being the least natural number such that $3t/n!$.

8. The partition "bai" of the odd numbers.

$$\text{Let } A = \{k \in \mathbf{N} | k \text{ odd and } F(k) = S(k)\}$$

$$B = \{k \in \mathbf{N} | k \text{ odd and } F(k) < S(k)\}$$

(A, B) is the partition "bai" of the odd numbers.
Into A there are numbers $k = 3t$, t odd, $t \ne 1$. Obviously, A has infinitly many elements.
Into B there are numbers $k = t! \pm 1$ with $t \ge 3$, $t \in \mathbf{N}$. Obviously, B has infinitly many elements.

Definition 1 *Let $n \in \mathbf{N}^*$. We called triplet $\hat{n}$, the set:*
$n-1, n, n+1$.

Definition 2 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are separated if*
$k+1 < n-1$, *i.e.* $n-k > 2$.

Definition 3 *The triplets $\hat{k}$, $\hat{n}$ are l_s-relatively prime if*
$(k-1, n-1) = 1$, $(k+1, n+1) \ne 1$.

For example: $\hat{6}$ and $\widehat{72}$ are l_s-relatively prime.

Definition 4 *The triplets $\hat{k}$, $\hat{n}$ are l_d-relatively prime if*
$(k-1, n-1) \ne 1$, $(k+1, n+1) = 1$.

Definition 5 *The triplets $\hat{k}$, $\hat{n}$ are l-relatively prime if*
$(k-1, n-1) = 1$, $(k+1, n+1) = 1$.

Definition 6 *The triplets $\hat{k}$, $\hat{n}$ are d-relatively prime if*
$(k-1, n+1) = 1,\ (k+1, n-1) = 1$.

For example: $\hat{2}$ and $\hat{6}$ are d-relatively prime.

Definition 7 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are d_s-relatively prime if*
$(k-1, n+1) = 1,\ (k+1, n-1) \neq 1$.

For example: $\hat{6}$ and $\widehat{120}$ are d_s-relatively prime.

Definition 8 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are d_d-relatively prime if*
$(k-1, n+1) \neq 1,\ (k+1, n-1) = 1$.

Example: $\hat{6}$ and $\widehat{24}$ are d_d-relatively prime.

Definition 9 *The triplets $\hat{k}$, $\hat{n}$ are p-relatively prime if*
$(k-1, n-1) = 1,\ (k-1, n+1) = 1,\ (k+1, n-1) = 1,\ (k+1, n+1) = 1$.

Obviously, if $\hat{k}$, $\hat{n}$ are p-relatively prime, then they are l and d-relatively prime.
For example: $\widehat{24}$ and $\widehat{120}$ are p-relatively prime.

Definition 10 *Let $k < n$. The triplets $\hat{k}$, $\hat{n}$ are F-relatively prime if*

$$(k-1, n-1) = 1,\ (k+1, n-1) = 1,$$
$$(k-1, n) = 1,\ (k+1, n) = 1$$
$$(k-1, n+1) = 1,\ (k+1, n+1) = 1.$$

Definition 11 *The triplets $\hat{k}$, $\hat{n}$ are t-relatively prime if*
$(k-1, n-1)\cdot(k-1, n)\cdot(k-1, n+1)\cdot(k, n-1)\cdot(k, n)\cdot(k, n+1)\cdot$
$(k+1, n-1)\cdot(k+1, n)\cdot(k+1, n+1) = 6$.

For example: $\hat{2}$ and $\hat{4}$ and t-relatively prime.

Definition 12 *Let $H \subset \mathbf{N}^*$. The triplet $\hat{n}$, $n \in H$ is, respectively, $l_s, l_d, l, d, d_s, d_d, p, F, t$-prime concerned at H, if $\forall s \in H$, $s < n$, the triplets $\hat{s}$, $\hat{n}$ are, respectively, $l_s, l_d, l, d, d_s, d_d, p, F, t$-relatively prime.*

Let $H = \{n! | n \in \mathbf{N}^*\}$. For the triplets $\hat{m}$, $m \in H$ there are particular properties.

Proposition 1 *Let $k < n$. The triplets $\widehat{(k!)}$, $\widehat{(n!)}$ are separated if*
$n > \max\{2, k\}$.

Proof. Obviously, $n! - k! > 2$ if $n > 2$ and $k < n$, i.e. $n > \max\{2, k\}$.

Proposition 2 *Let* $n > \max\{2, k\}$ *and* $M_{kn} = \{m \in \mathrm{N} | k!+1 < m < n!-1\}$. *If* $k_1 < k_2$ *and* $n_1 > \max\{2, k_1\}$, $n_2 > \max\{2, k_2\}$, *then* $n_1 - k_1 \leq n_2 - k_2 \Rightarrow card M_{k_1 n_1} < card M_{k_2 n_2}$.

Proof. For $n > k \geq 2$ it is true that

$$n! - (n-1)! > k! - (k-1)! \tag{1}$$

Let $n > k \geq 2$, $1 \leq s \leq k$. Using (1) we can write:

$$\begin{gathered} n! - (n-1)! > k! - (k-1)! \\ (n-1)! - (n-2)! > (k-1)! - (k-2)! \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (n-s-1)! - (n-s)! > (k-s-1)! - (k-s)! \end{gathered}$$

By summing this inequalities, it results:

$$n! - (n-s)! > k! - (k-s)! \tag{2}$$

Let $2 \leq k_1 < n_1$, $2 \leq k_2 < n_2$, $k_1 < k_2$, $n_1 - k_1 \leq n_2 - k_2$. Then $n_2 - n_1 \geq k_2 - k_1 \geq 1$ and there is n_3 such that $n_2 > n_3 \geq n_1$ and $n_2 - n_3 = k_2 - k_1$.
Using (2) we can write:
$n_2! - n_3! > k_2! - k_1!$
Since $n_3! \geq n_1!$ we have:

$$n_2! - n_1! > k_2! - k_1! \tag{3}$$

According to $card M_{k_1 n_1} = n_1! - 1 - (k_1! + 1)$,
$card M_{k_2 n_2} = n_2! - 1 - (k_2! + 1)$, it results that:

$$card M_{k_2 n_2} - card M_{k_1 n_1} = n_2! - n_1! - (k_2! - k_1!)$$

That is, taking into account (3), $card M_{k_1 n_1} < card M_{k_2 n_2}$.

Definition 13 *Let* $k < n$. *The triplets* $\widehat{(k!)}$, $\widehat{(n!)}$ *are linked if* $k! - 1 = n$ *or* $k! + 1 = n$.

Proposition 3 *For* $k \in \mathrm{N}^*$ *there is* p *prime number, such that for any* $s \geq p$ *the triplets* $\widehat{(k!)}$, $\widehat{(s!)}$ *are not* F*-relatively prime.*

Proof. Obviously, for $k = 1$ and $k = 2$, the proposition is true.
If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_i^{\alpha_i}$ divide $k! - 1$ or $k! + 1$, then $p_j > k \geq 3$, for $j \in \{1, 2, \ldots, i\}$.
Let $\bar{n} = p_1 \cdot p_2 \cdots p_i$ and $p = \max_{1 \leq j \leq i} \{p_j\}$.
Obviously, $\bar{n} \geq 3$ because $p > k \geq 3$, $\bar{n}/k! - 1$ or $\bar{n}/k! + 1$.
For any $s \geq p$, $\bar{n}/s!$ and so, the triplets $\widehat{(k!)}$, $\widehat{(s!)}$ are not F-relatively prime.

Remark 1 **i)** *Let $k < n$. If $\widehat{(k!)}$, $\widehat{(n!)}$ are linked, then $n - k = k! - k \pm 1$. If $2 < k_1 < n_1$, $\widehat{(k_1!)}$ with $\widehat{(n_1!)}$ are linked and $k_2 < n_2$, $\widehat{(k_2!)}$ with $\widehat{(n_2!)}$ are linked, then $k_1 < k_2 \Rightarrow n_1 - k_1 < n_2 - k_2$ and in view of the proposition 2, results $card M_{k_1 n_1} < card M_{k_2 n_2}$.*
ii) *There are twin prime numbers with the triplet $\widehat{(n!)}$. For example 5 with 7 are from $\widehat{(3!)}$.*

Definition 14 *Considering the canonical decomposition of natural numbers $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, we define $\tilde{n} = \{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}\}$, $\mathcal{M} = \{\tilde{n} | n \in \mathbf{N}^*\}$.*

Definition 15 *On $\mathcal{M}$ we consider the relation of order $\sqsubseteq$ defined by:*

$$\{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_r^{\alpha_r}\} \sqsubseteq \{q_1^{\beta_1}, q_2^{\beta_2}, \ldots, q_t^{\beta_t}\}$$

if and only if $\{p_1, p_2, \ldots, p_r\} \subset \{q_1, q_2, \ldots, q_t\}$ and if $p_i = q_j$, then $\alpha_i \leq \beta_j$.

Remark 2 *For any triplet $\widehat{(n!)}$, $n \in \mathbf{N}^*$, we consider the sets:*
$A_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n!}\}$, $A_n^* = \{k \in A_n | k \notin A_h \textit{ for } h < n\}$
$B_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n! - 1}\}$, $B_n^* = \{k \in B_n | k \notin B_h \textit{ for } h < n\}$
$C_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n! + 1}\}$, $C_n^* = \{k \in C_n | k \notin C_h \textit{ for } h < n\}$
$M_n = \{k \in \mathbf{N}^* | \tilde{k} \sqsubseteq \widetilde{n!} \textit{ or } \tilde{k} \sqsubseteq \widetilde{n! - 1} \textit{ or } \tilde{k} \sqsubseteq \widetilde{n! + 1}\}$
$M_n^* = \{k \in M_n | k \notin M_h \textit{ for } h < n\}$.
It is obvious that:
$A_n^* = S^{-1}(n)$, $B_n^* = f_s^{-1}(n)$, $C_n^* = f_d^{-1}(n)$, $M_n^* = F^{-1}(n)$.
If $k \in A_n^*$, it is said that k has a factorial signature which is equivalent with the factorial signature of $n!$ (see [1]).
Let $k \in B_n^*$, $k = t_1^{r_1} \cdot t_2^{r_2} \cdots t_i^{r_i}$. Then $\{t_r\} \not\sqsubseteq \widetilde{n!}$ for $r = \overline{1, i}$ and for any $h < n$, there are $t_j^{r_j}$, $1 \leq j \leq i$, such that $\{t_j^{r_j}\} \not\sqsubseteq \widetilde{h! - 1}$.
Similarly, for $k \in C_n^*$: $\{t_r\} \not\sqsubseteq \widetilde{n!}$ for $r = \overline{1, i}$ and for any $h < n$, there are $t_j^{r_j}$, $1 \leq j \leq i$, such that $\{t_j^{r_j}\} \not\sqsubseteq \widetilde{h! + 1}$.

References

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