ON THE k-POWER COMPLEMENT AND k-POWER FREE NUMBER SEQUENCE

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ABSTRACT. The main purpose of this paper is to study the distribution properties of k-power free numbers and k-power complement numbers, and give an interesting asymptotic formula.

1. INTRODUCTION AND RESULTS

Let $k \geq 2$ is a positive integer, a natural number n is called a k-power free number if it can not be divided by any p^k, where p is a prime number. One can obtain all k-power free number by the following method: From the set of natural numbers (except 0 and 1)

-take off all multiples of 2^k(i.e. 2^k, 2^{k+1}, 2^{k+2} ...).

-take off all multiples of 3^k.

-take off all multiples of 5^k.

...and so on (take off all multiples of all k-power primes).

For instance, the k-power free number sequence is called cube free sieve if $k = 3$, this sequence is the following $2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, \cdots$.

Let $n \geq 2$ is any integer, $a(n)$ is called a k-power complement about n if $a(n)$ is the smallest integer such that $n \times a(n)$ is a perfect k-power, for example $a(2) = 2^{k-1}$, $a(3) = 3^{k-1}$, $a(2^k) = 1, \cdots$.

In reference [1], Professor F. Smarandache asked us to study the properties of the k-power free number sequence and k-power complement number sequence. About these problems, it seems that none had studied them before. In this paper, we use the elementary method to study the distribution properties of these sequences, and obtain an interesting asymptotic formula. For convenience, we define $\Omega(n)$ and $\omega(n)$ as following: $\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r$, $\omega(n) = r$, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. Then we have the following Theorem.

Key words and phrases. k-power free numbers; k-power complement numbers, Mean Value; Asymptotic formula.

Theorem. *Let $\mathcal{A}$ denotes the set of all k-power free numbers. Then for any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \Omega(a(n)) = \frac{(k-1)x\ln\ln x}{\zeta(k)} + u(k)x + O\left(\frac{x}{\ln x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, $u(k)$ is a constant depending only on k .

2. Several lemmas

Lemma 1. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} \omega(n) = x\ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

$$\sum_{n\leq x} \Omega(n) = x\ln\ln x + Bx + O\left(\frac{x}{\ln x}\right).$$

where $A = \gamma + \sum_p \left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$, $B = A + \sum_p \frac{1}{p(p-1)}$.

Proof. (See reference [2]).

Lemma 2. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \omega(n) = \zeta^{-1}(k)x\ln\ln x + Ax\zeta^{-1}(k) + Cx + O\left(\frac{x}{\ln x}\right).$$

Proof. Let (u,v) denotes the greatest common divisor of u and v. Then from Lemma 1 we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \omega(n) = \sum_{n\leq x}\omega(n)\sum_{d^k|n}\mu(d) = \sum_{d^k n\leq x}\omega(nd^k)\mu(d) = \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{n\leq x/d^k}\omega(nd^k)$$

$$= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\left[\sum_{n\leq x/d^k}(\omega(n)+\omega(d)-\omega((n,d)))\right]$$

$$= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{n\leq x/d^k}\omega(n) + \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\omega(d)\left[\frac{x}{d^k}\right] - \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{\substack{n\leq x/d^k\\ u|n}}\omega(u)$$

$$= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\left[\frac{x}{d^k}\ln\ln\frac{x}{d^k} + \frac{Ax}{d^k} + O\left(\min\left(1, \frac{x}{d^k\ln\frac{x}{d^k}}\right)\right)\right]$$

$$+ x\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)\omega(d)}{d^k} + O\left(x^{\frac{1}{k}}\ln x\right) - \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\omega(u)\left[\frac{x}{ud^k}\right]$$

$$= x\ln\ln x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} + Ax\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} + x\sum_{d=1}^{\infty}\frac{\mu(d)\omega(d)}{d^k} - x\sum_{d=1}^{\infty}\frac{\mu(d)\sum_{u|d}\frac{\omega(u)}{u}}{d^k} + O\left(\frac{x}{\ln x}\right)$$

$$= \zeta^{-1}(k)x\ln\ln x + Ax\zeta^{-1}(k) + Cx + O\left(\frac{x}{\ln x}\right).$$

where

$$C = \sum_{d=1}^{\infty}\frac{\mu(d)\omega(d)}{d^k} - \sum_{d=1}^{\infty}\frac{\mu(d)\sum_{u|d}\frac{\omega(u)}{u}}{d^k}.$$

This proves Lemma 2.

Lemma 3. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\Omega(n) = \zeta^{-1}(k)x\ln\ln x + Bx\zeta^{-1}(k) + Dx + O\left(\frac{x}{\ln x}\right).$$

Proof. From Lemma 1, we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\Omega(n) = \sum_{n\leq x}\Omega(n)\sum_{d^k|n}\mu(d) = \sum_{d^k n\leq x}\Omega(nd^k)\mu(d) = \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{n\leq x/d^k}\Omega(nd^k)$$

$$= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\left[\sum_{n\leq x/d^k}(\Omega(n) + k\Omega(d))\right]$$

$$= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{n\leq x/d^k}\Omega(n) + \sum_{d\leq x^{\frac{1}{k}}}\mu(d)k\Omega(d)\left[\frac{x}{d^k}\right]$$

$$= \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\left[\frac{x}{d^k}\ln\ln\frac{x}{d^k} + \frac{Bx}{d^k} + O\left(\min\left(1, \frac{x}{d^k\ln\frac{x}{d^k}}\right)\right)\right]$$

$$+ kx\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)\Omega(d)}{d^k} + O\left(x^{\frac{1}{k}}\ln x\right)$$

$$= x\ln\ln x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} + Bx\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k} + kx\sum_{d=1}^{\infty}\frac{\mu(d)\Omega(d)}{d^k}$$

$$= \zeta^{-1}(k)x\ln\ln x + Bx\zeta^{-1}(k) + Dx + O\left(\frac{x}{\ln x}\right),$$

where

$$D = k\sum_{d=1}^{\infty}\frac{\mu(d)\Omega(d)}{d^k}.$$

This proves Lemma 3.

3. Proof of the Theorem

In this section, we shall complete the proof of the Theorem. According to the definition of k-power complement number and k-power free number, and applying Lemma 2, 3, we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\Omega(n\times a(n)) = k\sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\omega(n) = \sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\Omega(n) + \sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\Omega(a(n)).$$

or

$$\begin{aligned}
\sum_{\substack{n\le x\\ n\in\mathcal{A}}} \Omega(a(n)) &= k\sum_{\substack{n\le x\\ n\in\mathcal{A}}} \omega(n) - \sum_{\substack{n\le x\\ n\in\mathcal{A}}} \Omega(n)\\
&= k\left[\zeta^{-1}(k)x\ln\ln x + Ax\zeta^{-1}(k) + Cx + O\left(\frac{x}{\ln x}\right)\right]\\
&- \left[\zeta^{-1}(k)x\ln\ln x + Bx\zeta^{-1}(k) + Dx + O\left(\frac{x}{\ln x}\right)\right]\\
&= \frac{(k-1)x\ln\ln x}{\zeta(k)} + u(k)x + O\left(\frac{x}{\ln x}\right).
\end{aligned}$$

where

$$u(k) = \frac{kA-B}{\zeta(k)} + kC - D$$

This completes the proof of the Theorem .

REFERENCES

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