PROPERTIES OF THE NUMERICAL FUNCTION F_S

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In this paper are studied some properties of the numerical function $F_S(x): \mathbf{N} - \{0,1\} \to \mathbf{N}$ $F_S(x) = \sum_{\substack{0<p\le x \\ p \text{ prime}}} S_p(x)$, where $S_p(x) = S(p^x)$ is the Smarandache function defined in [4].

Numerical example: $F_S(5) = S(2^5) + S(3^5) + S(5^5)$; $F_S(6) = S(2^6) + S(3^6) + S(5^6)$.

It is known that: $(p-1)r + 1 \le S(p^r) \le pr$ so $(p-1)r < S(p^r) \le pr$.

Than

$$x(p_1 + p_2 + \cdots + p_{\pi(x)} - \pi(x)) < F_S(x) \le x(p_1 + p_2 + \cdots + p_{\pi(x)}) \qquad (1)$$

Where $\pi(x)$ is the number of prime numbers smaller or equal with x.

PROPOSITION 1: The sequence $T(x) = 1 \quad \log F_S(x) + \sum_{i=2}^{x} \frac{1}{F_S(i)}$ has limit $-\infty$.

Proof. The inequality $F_S(x) > x(p_2 + \cdots + p_{\pi(x)} - \pi(x))$ implies $-\log F_S(x) <$
$< -\log x(p_1 + p_2 + \cdots + p_{\pi(x)} - \pi(x)) < -\log x(\pi(x) p_1 - \pi(x)) = -\log x - \log \pi(x) - \log(p_1 - 1)$.

Than for $x=i$ the inequality (1) become:

$i(p_1 + \cdots + p_{\pi(i)} - \pi(i)) < F_S(i) \le i(p_1 + \cdots + p_{\pi(i)})$ so:

$$\frac{1}{F_S(i)} < \frac{1}{i(p_1 + \cdots + p_{\pi(i)} - \pi(i))} < \frac{1}{i(p_1 \pi(i) - \pi(i))} = \frac{1}{i\pi(i)(p_1 - 1)}$$

Than $T(x) < 1 - \log(x) - \log \pi(x) - \log(p_1 - 1) + \sum_{i=2}^{x} \frac{1}{i\pi(i)(p_1 - 1)}$

$p_1 = 2 \Rightarrow T(x) = 1 - \log x - \log \pi(x) + \sum_{i=2}^{x} \frac{1}{i\pi(i)}$

$\Rightarrow \lim_{x\to\infty} T(x) \le 1 - \lim_{x\to\infty} \log x - \lim_{x\to\infty} \log \pi(x) + \lim_{x\to\infty} \sum_{i=2}^{x} \frac{1}{i\pi(i)} = 1 - \infty - \infty + L = -\infty$.

PROPOSITION 2. The equation $F_S(x) = F_S(x+1)$ has no solution for $x \in \mathbf{N} - \{0,1\}$.

Proof. First we consider that $x+1$ is a prime number with $x > 2$. In the particular case $x = 2$ we obtain $F_S(2) = S(2^2) = 4$; $F_S(3) = S(2^3) + S(3^3) = 4 + 9 = 13$. So $F_2(2) < F_S(3)$.

Next we shall write the inequalities:

$$x(p_1+\dots+p_{\pi(x)} - \pi(x)) \ < \ F_S(x) \ \leq \ x(p_1+\dots+p_{\pi(x)}) \qquad (2)$$

$$(x+1)(p_1+\dots+p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) \ < \ F_S(x+1) \ \leq \ (x+1)(p_1+\dots+p_{\pi(x)} + p_{\pi(x+1)})$$

Using the reductio ad absurdum method we suppose that the equation $F_S(x) = F_S(x+1)$ has solution. From (2) results the inequalities

$$(x+1)(p_1+\dots+p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) \ < \ F_S(x+1) \ \leq \ x(p_1+\dots+p_{\pi(x)}) \qquad (3)$$

From (3) results that:

$$x(p_1+\dots+p_{\pi(x)}) - (x+1)(p_1+\dots+p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) > 0$$

$$x(p_1+\dots+p_{\pi(x)}) - x(p_1+\dots+p_{\pi(x)}) - xp_{\pi(x+1)} + x\pi(x+1) - p_1 - \dots - p_{\pi(x)} - p_{\pi(x+1)} +$$

$$+\pi(x+1) > 0.$$

But $p_{\pi(x+1)} > \pi(x+1)$ so the diference from above is negative for $x > 0$, and we obtained a contradiction. So $F_S(x) = F_S(x+1)$ has no solution for $x+1$ a prime number.

Next, we demonstrate that the equation $F_S(x) = F_S(x+1)$ has no solution for x and $x+1$ both composite numbers.

Let p be a prime number satisfing conditions $p > \frac{x}{2}$ and $p \leq x-1$. Such p exists according to Bertrand's postulate for every $x \in \mathbf{N} - \{0,1\}$. Than in the factorial of the number $p(x-1)$, the number p appears at least x times.

So, we have $S(p^x) \leq p(x-1)$.

But $p(x-1) < px + p - x$ (if $p > \frac{x}{2}$) and $px + p - x = (p-1)(x+1) + 1 \leq S(p^{x+1})$. Therefore $\exists\, p \leq x-1$ so that $S(p^x) < S(p^{x+1})$.

Than $F_S(x) = S(p_1^x)+\dots+S(p^x)+\dots+S(p_{\pi(x)}^x)$

$F_S(x+1) = S(p_1^{x+1})+\dots+S(p^{x+1})+\dots+S(p_{\pi(x)}^{x+1}) > F_S(x)$

In conclusion $F_S(x+1) > F_S(x)$ for x and $x+1$ composite numbers. If x is a prime number $\pi(x) = \pi(x+1)$ and the fact that the equation $F_S(x) = F_S(x+1)$ has no solution has the same demonstration as above.

Finally the equation $F_S(x) = F_S(x+1)$ has no solution for any $x \in \mathbf{N} - \{0,1\}$.

***PROPOSITION** 3.* The function $F_S(x)$ is strictly increasing function on its domain of definition.

The proof. of this property is justified by the proposition 2.

***PROPOSITION** 4.* $F_S(x+y) \ > \ F_S(x) + F_S(y) \quad \forall x,y \in \mathbf{N} - \{0,1\}$.

Proof. Let $x,y \in \mathbf{N} - \{0,1\}$ and we suppose $x < y$. According to the definition of $F_S(x)$ we have:

$$F(x+y)=S(p_1^{x+y})+\cdots+S(p_{\pi(x)}^{x+y})+S(p_{\pi(x)+1}^{x+y})+\cdots+S(p_{\pi(y)}^{x+y})+ \quad (4)$$

$$+S(p_{\pi(y)+1}^{x+y})+\cdots+S(p_{\pi(x+y)}^{x+y})$$

$$F(x)+F(y)=S(p_1^{x})+\cdots+S(p_{\pi(x)+1}^{x})+S(p_1^{y})+\cdots+S(p_{\pi(x)}^{y})+S(p_{\pi(x)+1}^{y})+\cdots+S(p_{\pi(y)}^{x})$$

But from (1) we have the following inequalities:

$$A=(x+y)(p_1+\cdots+p_{\pi(x)}+p_{\pi(x)+1}+\cdots+p_{\pi(x+y)}-\pi(x+y))<F(x+y)\leq$$

$$\leq(x+y)(p_1+\cdots+p_{\pi(x)}+p_{\pi(x)+1}+\cdots+p_{\pi(x+y)}) \quad (5)$$

and

$$x(p_1+\cdots+p_{\pi(x)}-\pi(x))+y(p_1+\cdots+p_{\pi(x)}+\cdots+p_{\pi(x)}+\cdots+p_{\pi(y)}-\pi(y)) < F(x)+F(y)\leq$$

$$\leq x(p_1+\cdots+p_{\pi(x)})+y(p_1+\cdots+p_{\pi(x)}+p_{\pi(x)+1}+\cdots+p_{\pi(y)})=B \quad (6)$$

We proof that B < A.

$$B<A \Leftrightarrow x(p_1+\cdots+p_{\pi(x)})+y(p_1+\cdots+p_{\pi(x)})+y(p_{\pi(x)+1}+\cdots+p_{\pi(y)}))<$$

$$x(p_1+\cdots+p_{\pi(x)})+y(p_1+\cdots-p_{\pi(x)})+x(p_{\pi(x)+1}+\cdots+p_{\pi(x+y)})-x\pi(x+y)+$$

$$+y(p_{\pi(x)+1}+\cdots+p_{\pi(y)})+y(p_{\pi(y)+1}+\cdots+p_{\pi(x+y)})-y\pi(x+y) \quad \Leftrightarrow$$

$$x(p_{\pi(x)+1}+\cdots+p_{\pi(x+y)}-\pi(x+y))+y(p_{\pi(y)+1}+\cdots+p_{\pi(x+y)}-\pi(x+y)) > 0$$

But $p_{\pi(x+y)}\geq\pi(x+y)$ so that the inequality from above is true.

CONSEQUENCE: $F_S(xy)>F_S(x)+F_S(y) \quad \forall\, x,y\in\mathbf{N}-\{0,1\}$
Because x and $y\in\mathbf{N}-\{0,1\}$ and $xy>x+y$ than $F_S(xy)>F_S(x+y)>F_S(x)+F_S(y)$

PROPOSITION 5. We try to find $\lim_{n\to\infty}\dfrac{F_S(n)}{n^{\alpha}}$

We have $F_S(n) = \sum_{\substack{0<p_i\leq n\\ p_i=prime}} S(p_i^n)$ and:

$$\frac{p_1+p_2+\cdots+p_{\pi(n)}-\pi(n)}{n^{\alpha-1}} < \frac{F_S(n)}{n^{\alpha}} \leq \frac{p_1+p_2+\cdots+p_{\pi(n)}}{n^{\alpha-1}}$$

If $\alpha<1$ than

$$\lim_{n\to\infty} n^{1-\alpha}(p_1+\cdots+p_{\pi(n)}-\pi(n))=\infty\cdot\infty=+\infty \quad\Rightarrow\quad \lim_{n\to\infty}\frac{F_S(n)}{n^{\alpha-1}}=+\infty.$$

If $\alpha=1$ than

$$\lim_{n\to\infty} n^{1-\alpha}(p_1+\cdots+p_{\pi(n)}-\pi(n))=\lim_{n\to\infty}(p_1+\cdots+p_{\pi(n)}-\pi(n))=+\infty\Rightarrow\lim_{n\to\infty}\frac{F_S(n)}{n^{\alpha-1}}=+\infty$$

We consider now $\alpha > 1$.

We try to find $\lim_{n\to\infty} \frac{\sum_{i=1}^{\pi(n)} p_i - \pi(n)}{n^{\alpha-1}}$ and $\lim_{n\to\infty} \frac{\sum_{i=1}^{\pi(n)} p_i}{n^{\alpha-1}}$ appling Stolz - Cesaro:

Let $a_n = \sum_{i=1}^{\pi(n)} p_i - \pi(n)$ and $b_n = n^{\alpha-1}$.

$$\text{Than}: \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\sum_{i=1}^{\pi(n+1)} p_i - \pi(n+1) - \sum_{i=1}^{\pi(n)} p_i + \pi(n)}{(n+1)^{\alpha-1} - n^{\alpha-1}} = \begin{cases} \dfrac{n}{(n+1)^{\alpha-1} - n^{\alpha-1}} \\ \text{if } (n+1) \text{ is a prime} \\ 0, \text{ otherwise} \end{cases}.$$

Let $c_n = \sum_{i=1}^{\pi(n)} p_i$ and $d_n = n^{\alpha-1}$.

$$\text{Than } \frac{c_{n+1}-c_n}{d_{n+1}-d_n} = \frac{\sum_{i=1}^{\pi(n+1)} p_i - \sum_{i=1}^{\pi(n)} p_i}{(n+1)^{\alpha-1} - n^{\alpha-1}} = \frac{p_{\pi(n+1)}}{(n+1)^{\alpha-1} - n^{\alpha-1}} = \begin{cases} \dfrac{n+1}{(n+1)^{\alpha-1} - n^{\alpha-1}} \text{ if} \\ (n+1) \text{ is a prime} \\ 0, \text{ otherwise} \end{cases}.$$

First we consider the limit of the function.

$$\lim_{x\to\infty} \frac{x}{(x+1)^{\alpha-1} - x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{(\alpha-1)[(x+1)^{\alpha-2} - x^{\alpha-2}]} = 0 \quad \text{for } \alpha - 2 > 1$$

We used the l'Hospital theorem:
In the same way we have

$$\lim_{x\to\infty} \frac{x+1}{(x+1)^{\alpha-1} - x^{\alpha-1}} = 0 \quad \text{for } \alpha > 3.$$

So, for $\alpha > 3$ we have:

$$\lim_{x\to\infty} \frac{p_1 + p_2 + \cdots + p_{\pi(n)} - \pi(n)}{n^{\alpha-1}} = 0 \text{ and}$$

$$\lim_{x\to\infty} \frac{p_1 + p_2 + \cdots + p_{\pi(n)}}{n^{\alpha-1}} = 0. \quad \text{So } \lim_{x\to\infty} \frac{F(n)}{n^{\alpha}} = 0.$$

$$\text{Finally } \lim_{x\to\infty} \frac{F(n)}{n^{\alpha}} = \begin{cases} 0 & \text{for } \alpha > 3 \\ +\infty & \text{for } \alpha \le 1 \end{cases}$$

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