ON THE K-POWER PART RESIDUE FUNCTION

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the asymptotic properties of the k-power part residue, and give an interesting asymptotic formula for it.

Keywords: k-power part residues function; Asymptotic formula.

§1. Introduction and results

For any positive integer n, the Smarandache k-th power complements $b_k(n)$ is the smallest positive integer such that $nb_k(n)$ is a complete k-th power (see probem 29 of [1].) Similar to the Smarandache k-th power complements, the additive k-th power complements $a_k(n)$ is defined as the smallest nonnegative integer such that $a_k(n)+n$ is a complete k-th power. About this problem, some authors had studied it, and obtained some interesting results. For example, in [4] Xu Z.F. used the elementary method to study the mean value properties of $a_k(n)$ and $d(a_k(n))$. in [5] Yi Y. and Liang F.C. used the analytic method to study the mean value properties of $d(a_2(n))$, and obtained a sharper asymptotic formula for it.

Similarly, we will define the k-power part residue function as following: For any positive integer n, it is clear that there exists a positive integer N such that $N^k \leq n < (N+1)^k$. Let $n = N^k + r$, then $f_k(n) = r$ is called the k-power part residue of n. In this paper, we use the elementary and analytic methods to study the asymptotic properties of this sequence, and obtain two interesting asymptotic formulae for it. That is, we shall prove the following:

Theorem. *For any real number $x > 1$ and any fixed positive integer m and k, we have the asymptotic formula*

$$\sum_{n\leq x} \delta_m(f_k(n)) = \frac{k^2}{2(2k-1)} \prod_{p|m} \frac{p}{p+1} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right),$$

where $\prod_{p|m}$ denotes the product over all prime divisors of m, and

$$\delta_m(n) = \begin{cases} max\{d \in N \quad | \quad d|n, (d,m) = 1\}, & if \quad n \neq 0, \\ 0, & if \quad n = 0. \end{cases}$$

Especially taking $m = 1$, and note that $\delta_1(f_k(n)) = f_k(n)$ we may immediately get the following:

Corollary . *For any real number $x > 1$ and any fixed positive integer k, we have the asymptotic formula*

$$\sum_{n \leq x} f_k(n) = \frac{k^2}{2(2k-1)} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

§2. Proof of Theorem

In this section, we will complete the proof of Theorem. First we need following

Lemma. *For any real number $x > 1$ and positive integer m, we have*

$$\sum_{n \leq x} \delta_m(n) = \frac{x^2}{2} \prod_{p|k} \frac{p}{p+1} + O(x^{\frac{3}{2}+\epsilon}),$$

where ϵ is any positive number.

Proof. Let $s = \sigma + it$ be a complex number and $f(s) = \sum\limits_{n=1}^{\infty} \frac{\delta_m(n)}{n^s}$. Note that $\delta_m(n) \ll n$, so it is clear that $f(s)$ is an absolutely convergent series for Re(s)> 2, by the Euler product formula [2] and the definition of $\delta_m(n)$ we get

$$\begin{aligned} f(s) = \sum_{n=1}^{\infty} \frac{\delta_m(n)}{n^s} &= \prod_p \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \\ &= \prod_{p|m} \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \\ &\quad \times \prod_{p \dagger m} \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \\ &= \prod_{p|m} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{ns}} + \cdots\right) \\ &\quad \times \prod_{p \dagger m} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{2n}}{p^{ns}} + \cdots\right) \\ &= \prod_{p|m} \left(\frac{1}{1 - \frac{1}{p^s}}\right) \prod_{p \dagger m} \left(\frac{1}{1 - \frac{1}{p^{s-1}}}\right) \end{aligned}$$

$$= \quad \zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod_p$ denotes the product over all primes.

From (1) and Perron's formula [3], we have

$$\sum_{n\leq x}\delta_m(n)=\frac{1}{2\pi i}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds+O\left(\frac{x^{\frac{5}{2}+\epsilon}}{T}\right), \tag{2}$$

where ϵ is any positive number.

Now we move the integral line in (2) from $s=\frac{5}{2}\pm iT$ to $s=\frac{3}{2}\pm iT$. This time, the function $\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}$ has a simple pole point at $s=2$ with residue

$$\frac{x^2}{2}\prod_{p|m}\frac{p}{p+1}. \tag{3}$$

Hence, we have

$$\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}+\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}+\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}\right)\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds$$
$$=\frac{x^2}{2}\prod_{p|m}\frac{p}{p+1}. \tag{4}$$

We can easily get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}+\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}\right)\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds\right|\ll\frac{x^{\frac{5}{2}+\epsilon}}{T}, \tag{5}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds\right|\ll x^{\frac{3}{2}+\epsilon}. \tag{6}$$

Taking $T=x$, combining (2), (4), (5) and (6) we deduce that

$$\sum_{n\leq x}\delta_m(n)=\frac{x^2}{2}\prod_{p|m}\frac{p}{p+1}+O(x^{\frac{3}{2}+\epsilon}). \tag{7}$$

This completes the proof of Lemma.

Now we shall use the above lemma to complete the proof of Theorem. For any real number $x\geq 1$, let M be a fixed positive integer such that

$$M^k\leq x<(M+1)^k. \tag{8}$$

Then from (7) and the definition of $f_k(n)$, we have

$$\begin{aligned}
&\sum_{n\le x}\delta_m(f_k(n)) \qquad (9)\\
=\ &\sum_{t=1}^{M}\sum_{(t-1)^k\le n<t^k}\delta_m(f_k(n))+\sum_{M^k\le n<x}\delta_m(f_k(n))\\
=\ &\sum_{t=1}^{M-1}\sum_{t^k\le n<(t+1)^k}\delta_m(f_k(n))+\sum_{M^k\le n\le x}\delta_m(f_k(n))\\
=\ &\sum_{t=1}^{M}\sum_{j=0}^{(t+1)^k-t^k}\delta_m(j)+O\left(\sum_{M^k\le n<(M+1)^k}\delta_m(f_k(n))\right)\\
=\ &\sum_{t=1}^{M}\left(\frac{\left((t+1)^k-t^k\right)^2}{2}\prod_{p|m}\frac{p}{p+1}+O\left(\left((t+1)^k-t^k\right)^{\frac{3}{2}+\epsilon}\right)\right)+O\left(M^k\right)\\
=\ &\frac{1}{2}\prod_{p|m}\frac{p}{p+1}\left(\sum_{t=1}^{M}\left((t+1)^k-t^k\right)^2\right)+O\left(\sum_{t=1}^{M}t^{(k-1)(\frac{3}{2}+\epsilon)}\right)\\
=\ &\frac{k^2}{2}\prod_{p|m}\frac{p}{p+1}\sum_{t=1}^{M}t^{2(k-1)}+O\left(\sum_{t=1}^{M}t^{2k-3}\right)\\
=\ &\frac{k^2M^{2k-1}}{2(2k-1)}\prod_{p|m}\frac{p}{p+1}+O\left(M^{2k-2}\right). \qquad (10)
\end{aligned}$$

On the other hand, we also have the estimate

$$0\le x-M^k<(M+1)^k-M^k\ll x^{\frac{k-1}{k}}.$$

Now combining (9) and (10) we may immediately obtain the asympotic formula

$$\sum_{n\le x}\delta_m(f_k(n))=\frac{k^2}{2(2k-1)}\prod_{p|m}\frac{p}{p+1}x^{2-\frac{1}{k}}+O\left(x^{2-\frac{2}{k}}\right).$$

This completes the proof of Theorem.

References

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