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A Note on the Maximum Genus of Graphs with Diameter 4

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Abstract: Let G be a simple graph with diameter four,if G does not contain complete subgraph K_3 of order three. We prove that the Betti deficient number of G, $\xi(G) \leq 2$. i.e. the maximum genus of G, $\gamma_M(G) \geq \frac{1}{2}\beta(G) - 1$ in this paper, which is related with *Smarandache 2-manifolds* with minimum faces.

Key words: Diameter, Betti deficiency number, maximum genus.

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§1. Preliminaries and known results

In this paper, G is a finite undirected simple connected graph. The maximum genus $\gamma_M(G)$ of G is the largest genus of an orientable surface on which G has a 2-cell embedding, and $\xi(G)$ is the Betti deficiency of G. To determine the maximum genus $\gamma_M(G)$ of a graph G on orientable surfaces is related with *map geometries*, i.e., *Smarandache 2-manifolds* (see [1] for details) with minimum faces.

By Xuong's theory on the maximum genus of a connected graph, $\xi(G)$ equal to $\beta(G) - 2\gamma_M(G)$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the Betti number of G. For convenience, we use *deficiency* to replace the words *Betti deficiency* in this paper. Nebeský[2] showed that if G is a connected graph and $A \subseteq E(G)$, let $\upsilon(G, A) = c(G - A) + b(G - A) - |A| - 1$, where $c(G - A)$ denotes the number of components in $G - A$ and $b(G - A)$ denotes the number of components in $G - A$ with an odd Betti number, then we have $\xi(G) = max\{\upsilon(G, A)|A \subseteq E(G)\}$.

Clearly, the maximum genus of a graph can be determined by its deficiency. In case of that $\xi(G) \leq 1$, the graph G is said to be upper embeddable. As we known, following theorems are the main results on relations of the maximum genus with diameter of a graph.

Theorem 1.1 *Let G be a multigraph of diameter 2. Then $\xi(G) \leq 1$.*

Skoviera proved Theorem 1.1 by a different method in [3] − [4].

Hunglin Fu and Minchu Tsai considered multigraphs of diameter 3 and proved the following theorem in [5].

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Theorem 1.2 *Let G be a multigraph of diameter 3. Then $\xi(G) \leq 2$.*

When the diameter of graphs is larger than 3, the Betti deficiency of G is unbounded. The following investigations have focused on graphs with a given diameter and some characters. Some results in this direction are presented in the following.

Theorem 1.3([16]) *Let G be a 3-connected multigraph of diameter 4, then $\xi(G) \leq 4$.*

Theorem 1.4([16]) *Let G be a 3-connected simple graph of diameter 5. Then $\xi(G) \leq 18$.*

Yuanqiu Huang and Yanpei Liu proved the following result in [6].

Theorem 1.5 *Let G be a simple, K_3-free graph of diameter 4, then $\xi(G) \leq 4$, where K_3-free graph means that there are no spanning subgraphs K_3 in G.*

The main purpose of this paper is to improve this result.

§2. Main result and its proof

Nebeský's method is useful and the minimality property of the edge subset A in this method plays an important role. For convenience, we call a graph with $\xi(G) \geq 2$ a *deficient graph*. Any set $A \subseteq E(G)$ such that $\upsilon(G, A) = \xi(G)$ will be called a *Nebeský set*. Furthermore, if A is minimal, then it will be called a *minimal Nebeský set*.

Lemma 2.1([5]) *Let G be a deficient graph and A a minimal Nebeský set of G. Then*

(*a*) *$b(G-A) = c(G-A) \geq 2$. More, if G is a simple graph then every component of $G-A$ contains at least three vertices;*

(*b*) *the end vertices of every edge in A belong to distinct components of $G-A$;*

(*c*) *any two components of $G-A$ are joined by at most one edge of A;*

(*d*) *$\xi(G) = 2c(G-A) - |A| - 1$.*

With the support of Lemma 2.1, we are able to construct a new graph based on the choice of A. Let G be a deficient graph and A a minimal Nebeský set of G. G_A is called a *testable graph of* G if $V(G_A)$ is the set of components of $G-A$ and two vertices in G_A are adjacent if and only if they are joined in G by an edge of A. We shall refer the vertices of G_A to as the *nodes* of G_A, and u_A, υ_A,...are typical notation for the nodes.

Lemma 2.2 *Let G be a deficient graph and A a minimal Nebeský set of G. Then*

$$\xi(G) = 2p(G_A) - q(G_A) - 1,$$

where $p(G_A)$ and $q(G_A)$ are the numbers of nodes and edges of G_A, respectively.

Proof By the definition of G_A, we know that $p(G_A) = c(G-A)$ and $q(G_A) = |A|$. Applying Lemma 2.1, we find that

$$\xi(G) = 2c(G-A) - |A| - 1 = 2p(G_A) - q(G_A) - 1.$$

□

Lemma 2.3 *If G is triangle-free, there exist a $\omega_A \in V(G_A)$ such that $2 \leq |E(\omega_A, A)| \leq 3$, where $E(\omega_A, A)$ denotes the set of edges of G_A incident with ω_A.*

Proof Let T_{ω_A} denote the component of $G-A$ which corresponds to ω_A in G_A. By Lemma 2.1 $|V(G_A)| \geq 2$. If for all $\omega_A \in V(G_A)$, there is $|E(\omega_A, A)| \geq 4$, then

$$|A| = \frac{1}{2}\sum_{\omega_A \in V(G_A)} |E(\omega_A, A)| \geq 2|V(G_A)|.$$

Applying Lemma 2.1 and the definition of G_A, $\xi(G) = 2V(G_A) - |A| - 1 \leq -1$, a contradiction.

For G is connected, $|E(\omega_A, A)| \geq 1$. If $|E(\omega_A, A)| = 1$, let $E(\omega_A, A) = \{e\}$, $e = fh, f \in V(T_{\omega_A}), h \in V(T_{\sigma_A}), \sigma_A \in V(G_A)$. By Lemma 2.1, $\beta(T_{\omega_A})$ is odd and T_{ω_A} is simple and triangle-free, there exists $f' \in V(T_{\omega_A})$ such that $f' \neq f, ff' \notin E(G)$. Similarly, there exists $h' \in V(T_{\sigma_A})$ such that $h' \neq h, hh' \notin E(G)$. Since e is a bridge, $d_G(f', h') \geq 5$, a contradiction.

So we get that $2 \leq |E(\omega_A, A)| \leq 3$. □

Theorem *Let G be a simple, triangle-free graph of diameter 4, then $\xi(G) \leq 2$, i.e., the maximum genus of G, $\gamma_M(G) \geq \frac{1}{2}\beta(G) - 1$.*

Proof Let Π=$\{H|H$ is a simple graph of diameter 4 and does not contain a spanning subgraph K_3 with $\xi(G) > 2$ $\}$. We claim that Π is an empty set. Suppose it is not true, let $G \in \Pi$ be with minimum order. Clearly, G is a deficient graph. Now let A be a minimal Nebeský set. Applying Lemma 2.1(a), each component of $G - A$ has odd Betti number. Thus, each component of $G - A$ must be a quadrangle. Otherwise, there exists a graph $|V(G')| < |V(G)|$. Now let T_{x_A} denote the component of $G - A$ which corresponds to x_A in G_A for each node $x_A \in V(G_A)$.

By Lemma 2.3, choose $z_A \in V(G_A)$ with $2 \leq |E(z_A, A)| \leq 3$, and define $D_0 = \{z_A\}, D_1 = N(z_A)$ and $D_2 = V(G_A) - N(z_A)$. We call $x \in V(G)$ a distance k vertex, if min $\{d(x, z)|z \in V(T_{z_A})\} = k$ and denote $E(D_i, D_j) = \{x_Ay_A \in E(G_A)|x_A \in D_i$ and $y_A \in D_j\}$, where $0 \leq i, j \leq 2$ (Note that the order of x_A and y_A is important throughout of the proof). We also need the following definitions.

A_1=$\{x_Ay_A \in E(D_2, D_1)|$ there exists a distance 1 vertex of T_{y_A} adjacent to a distance 2 vertex of T_{x_A}, or a distance 2 vertex of T_{y_A} adjacent to a distance 3 vertex of T_{x_A} and a distance 1 vertex of T_{ω_A} for some $\omega_A \in D_1 - \{y_A\}\}$.

A_2=$\{x_Ay_A \in E(D_2, D_2)|$ x_A is not incident with any edge of A_1 and y_A is incident with one edge of A_1 and T_{y_A} contains a vertex both adjacent to a vertex of T_{x_A} and a vertex of T_{u_A} for some $u_A \in D_1\} \cup \{x_Ay_A \in E(D_2, D_2)|$ x_A is not incident with any edge of A_1 and y_A is incident with at least two edges of $A_1\}$.

A_3=$\{x_Ay_A \in E(D_1, D_1)|$ there exists a distance 2 vertex of T_{x_A} adjacent to a distance 1 vertex of $T_{y_A}\}$.

Now, according to these edge subsets $A_1 - A_3$ of $E(G_A)$, we define a directed graph $\overrightarrow{G_A}$ based on G_A:

(i) $V(\overrightarrow{G_A}) = V(G_A)$;

(ii) if $x_Ay_A \in E' = (\bigcup_{i=1}^{3} A_i) \bigcup (D_1, D_0)$, then join two arcs from y_A to x_A;

(iii) if $x_Ay_A \in E(G_A) - E^{'}$, then let (x_A, y_A) and (y_A, x_A) be arcs of $\overrightarrow{G_A}$.

By this definition, it is easy to see that

$$\sum_{x_A \in V(G_A)} deg(x_A) = \sum_{x_A \in V(\overrightarrow{G_A})} deg^{-}(x_A),$$

where $deg^{-}(x_A)$ denotes the in-degree of x_A in $\overrightarrow{G_A}$. Therefore, the in-degree sum of $\overrightarrow{G_A}$ gives $2q(G_A)$.

Now, we count the in-degree sum of $\overrightarrow{G_A}$. Let x_A be an arbitrary node in $V(\overrightarrow{G_A})$.

(1) $x_A \in D_0$. Then $deg^{-}(x_A) = 0$ clearly.

(2) $x_A \in D_2$. The situation is divided into the discussions (i)-(iv) following.

(i) x_A is not incident with edges of A_1, but incident with edges of A_2.

Case 1 x_A is incident with at least two edges of A_2, then $deg^{-}(x_A) \geq 4$.

Case 2 x_A is incident with one edge e of A_2. Let x_1y_1 be an edge of $E(G)$ which corresponds to the edge e. Accordingly, T_{z_A} is a quadrangle and $2 \leq |E(z_A, A)| \leq 3$. Then there exist $z_1 \in V(T_{z_A})$ and $deg(z_1) = 2$. We know that $d(x_1, z_1) = 4$ in G. Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$. In T_{x_A}, x_2 must be incident with an edge of $E(G_A) - E^{'}$ such that $d(x_2, z_1) \leq 4$(in fact $d(x_2, z_1) = 4$). Similar discussion can be done done for vertices x_3 and x_4. So $deg^{-}(x_A) \geq 4$.

(ii) x_A is not incident with edges of $A_1 \bigcup A_2$.

Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$. In T_{x_A}, x_1 must be incident with an edge of $E(G_A) - E^{'}$ such that $d(x_1, z_1) \leq 4$(in fact $d(x_2, z_1) = 4$). Similar discussion can be done done for vertices x_2,x_3 and x_4. So $deg^{-}(x_A) \geq 4$.

(iii) x_A is incident with edges of A_1, but not incident with edges of A_2.

Case 1 x_A is incident with at least two edges of A_1, then $deg^{-}(x_A) \geq 4$.

Case 2 x_A is incident with one edge e of A_1. Let x_1y_1 be an edge of $E(G)$ which corresponds to the edge e. Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$ and $d(x_1, z_1) \geq 3$. In T_{x_A}, it supposes that x_3 is not incident with x_1, then x_3 must be incident an edge of $E(G_A)$. Let this edge be $e^{'}$. Then $e^{'} \in E(G_A) - E^{'}$, and $e^{'}$ contributes one de-agree. So $deg^{-}(x_A) \geq 3$(in fact, when $deg^{-}(x_A) = 3$, $e^{'} \in E(D_2, D_1)$).

(iv) x_A is incident with edges of A_1 and A_2.

Case 1 x_A is incident with at least two edges of A_1, then $deg^{-}(x_A) \geq 4$.

Case 2 x_A is incident with one edge e of A_1. Let x_1y_1 be an edge of $E(G)$ which corresponds to the edge e. Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$ and $d(x_1, z_1) \geq 3$. In T_{x_A}, it supposes that x_3 is not incident with x_1, then x_3 must be incident with an edge of $E(G_A)$. Let this edge be $e^{'}$. Then $e^{'} \in A_2$ or $e^{'} \in E(G_A) - E^{'}$. In the former, $e^{'}$ must contributes two de-agree for x_A. In the latter, $e^{'}$ contributes one de-agree. So $deg^{-}(x_A) \geq 3$(in fact,when $deg^{-}(x_A) = 3$, $e^{'} \in E(D_2, D_1)$).

Hence, for $x_A \in D_2$, $deg^{-}(x_A) \geq 3$.

Let M=$\{x_A \in D_2 | deg^-(x_A) = 3\}$. We get that

$$\sum_{x_A \in D_2} deg^-(x_A) \geq 4|D_2| - |M|.$$

(3) $x_A \in D_1$.

By the definition of $\overrightarrow{G_A}$, the edge connects D_0 and D_1 contributes two de-agree for x_A. Let x_1y_1 be an edge of $E(G)$ corresponds to this edge($y_1 \in E(T_{z_A})$).

Let $V(T_{x_A}) = \{x_1, x_2, x_3, x_4\}$. In T_{x_A}, it supposes that x_3 is not incident with x_1. In T_{z_A}, there exists $z_2 \in V(T_{z_A})$ so that if $d(x_3, z_2) \leq 4$. If x_3 does not connect z_2 though x_2 or x_4, x_3 must be incident with one edge of $E(G_A)$. Let that edge be e. Then $e \in E(G_A) - E^{'}$ and $deg(x_A) \geq 3$. If x_3 connects z_2 though x_2 or x_4, x_2 or x_4 is incident with one edge of $E(G_A)$. Let that edge be e. Then $e \in E(G_A) - E^{'}$ or $e \in A_3$, and e contributes at least one de-agree. So $deg(x_A) \geq 3$.

Hence, for all $x_A \in D_1$,

$$\sum_{x_A \in D_1} deg^-(x_A) \geq 3|D_1| + |M|.$$

Now by discussions (1) and (2), we get that

$$\begin{aligned} 2q(G_A) &= \sum_{x_A \in V(\overrightarrow{G}_A)} deg^-(x_A) \\ &\geq 4|D_2| - |M| + 3|D_1| + |M| \\ &= 4p(G_A) - |D_1| - 4 \\ &\geq 4P(G_A) - 7. \end{aligned}$$

Applying Lemma 2.2 again, we get that $\xi(G) = 2p(G_A) - 1 - q(G_A) \leq 2$, also a contradiction. This completes the proof. $\square$

To see that the upper bound presented in our theorem is best possible, let us consider the following family of infinite graphs, as depicts in Fig. 1. There are even paths with length 2 from m to n. Thus, this graph is triangle-free with diameter 4. It is not difficult to check that its Betti deficiency are equal to 2.

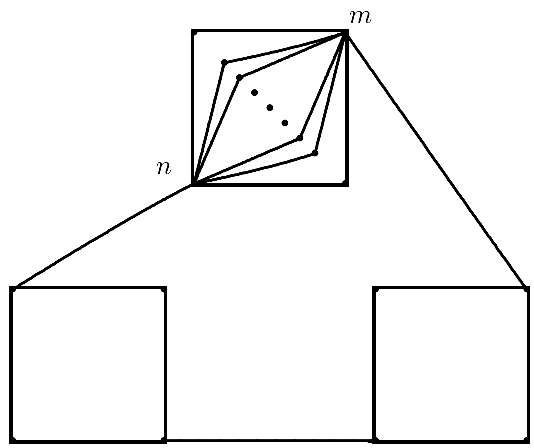

Fig.1

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