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Mean value of a Smarandache-Type Function

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Abstract In this paper, we use analytic method to study the mean value properties of Smarandache-Type Multiplicative Functions $K_m(n)$, and give its asymptotic formula . Finally, the convolution method is used to improve the error term.

Keywords Smarandache-Type Multiplicative Function, the Convolution method.

§1. Introduction

Suppose $m \geq 2$ is a fixed positive integer. If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, we define

$$K_m(n) = p_1^{\beta_1} p_2^{\beta_2} ... p_k^{\beta_k}, \quad \beta_i = \mathtt{min}(\alpha_i, m-1),$$

which is a Smarandache-type multiplicative function . Yang Cundian and Li Chao proved in [1] that

$$\sum_{n \leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O(x^{\frac{3}{2}+\epsilon}).$$

In this paper, we shall use the convolution method to prove the following

Theorem. The asymptotic formula

$$\sum_{n \leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O(x^{1+\frac{1}{m}} e^{-c_0 \delta(x)})$$

holds, where c_0 is an absolute positive constant and $\delta(x) = (\log x)^{3/5} (\log \log x)^{-1/5}$.

§2. Proof of the theorem

In order to prove our Theorem, we need the following Lemma, which is Lemma 14.2 of [2].

Lemma. Let $f(n)$ be an arithmetical function for which :

$$\sum_{n \leq x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a),$$

$$\sum_{n \leq x} \mid f(n) \mid = O(x^{a_1} \log^r x),$$

where $a_1 \geq a_2 \geq \ldots \geq a_l > 1/k > a \geq 0, r \geq 0, P_1(t), \cdots, P_l(t)$ are polynomials in t of degrees not exceeding r, and $k \geq 1$ is a fixed integer. If

$$h(n) = \sum_{d^k|n} \mu(d) f(n/d^k),$$

then

$$\sum_{n \leq x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E(x),$$

where $R_1(t), \ldots, R_l(t)$ are polynomials in t of degrees not exceeding r, and for some $D > 0$

$$E(x) \ll x^{1/k} exp\Big(- D(\log x)^{3/5} (\log\log x)^{-1/5}\Big).$$

Now we prove our Theorem. Let

$$g(s) = \sum_{n=1}^{\infty} \frac{K_m(n)}{n^s}, \Re(s) > 2.$$

According to Euler's product formula, we write

$$\begin{aligned}
g(s) \quad &= \prod_p \left(1 + \frac{K_m(p)}{p^s} + \frac{K_m(p^2)}{p^{2s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{K_m(p)}{p^s} + \frac{(K_m(p^2))}{p^{2s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{m-1}}{p^{(m-1)s}} + \frac{p^{m-1}}{p^{ms}} + \frac{p^{m-1}}{p^{(m+1)s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^{s-1}} + \frac{1}{p^{2(s-1)}} + \cdots + \frac{1}{p^{(m-1)(s-1)}} + \frac{p^{m-1}}{p^{ms}} + \frac{p^{m-1}}{p^{(m+1)s}} + \cdots\right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} + \frac{p^{m-1}}{p^{ms}} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots)\right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} + \frac{p^{m-1}}{p^{ms}} \frac{1}{1 - \frac{1}{p^s}}\right) \\
&= \prod_p \frac{1 - \frac{1}{p^{m(s-1)}}}{1 - \frac{1}{p^{s-1}}} \left(1 + \frac{p^{s-1} - 1}{(p^s - 1)(p^{m(s-1)} - 1)}\right) \\
&= \frac{\zeta(s-1)}{\zeta\big(m(s-1)\big)} R(s),
\end{aligned}$$

where

$$R(s) = \prod_p \left(1 + \frac{p^{s-1} - 1}{(p^s - 1)(p^{m(s-1)} - 1)}\right).$$

Let $q_m(n)$ denote the characteristic function of m-free numbers, then

$$\frac{\zeta(s)}{\zeta(ms)} = \sum_{n=1}^{\infty} \frac{q_m(n)}{n^s}, \quad \frac{\zeta(s-1)}{\zeta(m(s-1))} = \sum_{n=1}^{\infty} \frac{q_m(n)n}{n^s}.$$

Suppose

$$R(s) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s},$$

then

$$K_m(n) = \sum_{n=l_1 l_2} q_m(l_1) l_1 r(l_2).$$

Obviously, when $\sigma > 1$, R(s) absolutely converges, namely

$$\sum_{l \le x} \mid r(l) \mid \ll x^{1+\varepsilon}. \tag{1}$$

We can write $q_m(n)$ as the following form

$$q_m(n) = \sum_{d^k \mid n} \mu(d)$$

Now we apply the lemma on taking $f(n) = 1, \quad l = a_1 = 1, r = a = 0$, then we have

$$\sum_{n \le x} q_m(n) = \frac{x}{\zeta(m)} + O\left(x^{\frac{1}{m}} e^{-c_1 \delta(x)}\right)$$

for some absolute constant $c_1 > 0$.

By partial summation,

$$\sum_{n \le x} q_m(n) n = \frac{x^2}{2\zeta(m)} + O(x^{1+\frac{1}{m}} e^{-c_2 \delta(x)}) \tag{2}$$

holds for some absolute constant $c_2 > 0$. Let $y = x^{1-1/2m}$. By hyperbolic summation , we write

$$\begin{aligned} \sum_{n \le x} K_m(n) &= \sum_{l_1 l_2 \le x} q_m(l_1) l_1 r(l_2) \qquad (3) \\ &= \sum_{l_2 \le y} r(l_2) \sum_{l_1 \le \frac{x}{l_2}} q_m(l_1) l_1 + \sum_{l_1 \le \frac{x}{y}} q_m(l_1) l_1 \sum_{l_1 \le \frac{x}{l_1}} r(l_2) - \sum_{l_2 \le y} r(l_2) \sum_{l_1 \le \frac{x}{y}} q_m(l_1) l_1 \\ &= \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3 . \end{aligned}$$

From (1) we get

$$\sum\nolimits_2 \ll \sum_{l_1 \le \frac{x}{y}} l_1 \left(\frac{x}{l_1}\right)^{1+\varepsilon} \ll \frac{x^{2+\varepsilon}}{y} \ll x^{1+1/2m+\varepsilon}. \tag{4}$$

Similarly

$$\sum\nolimits_3 \ll \frac{x^{2+\varepsilon}}{y} \ll x^{1+1/2m+\varepsilon}. \tag{5}$$

Finally for $\sum_1$ we have by (2)

$$\begin{aligned}\sum\nolimits_1 &= \frac{x^2}{2\zeta(m)}\sum_{l_2\le y}\frac{r(l_2)}{l_2^2} + O\Bigg(\sum_{l_2\le y} x^{1+\frac{1}{m}} l_2^{-1-\frac{1}{m}} e^{-c_2\delta\left(\frac{x}{l_2}\right)}\Bigg) \qquad (6)\\ &= \frac{x^2}{2\zeta(m)}R(2) + O\Bigg(x^2\sum_{l_2>y}\frac{r(l_2)}{l_2^2}\Bigg) + O\Big(x^{1+\frac{1}{m}} e^{-c_0\delta\left(x\right)}\Big)\\ &= \frac{x^2}{2\zeta(m)}R(2) + O\Bigg(\frac{x^{2+\varepsilon}}{y}\Bigg) + O\Big(x^{1+\frac{1}{m}} e^{-c_0\delta\left(x\right)}\Big)\\ &= \frac{x^2}{2\zeta(m)}R(2) + O\Big(x^{1+\frac{1}{m}} e^{-c_0\delta(x)}\Big),\end{aligned}$$

if we noticed that

$$\sum_{l_2>y}\frac{r(l_2)}{l_2^2} \ll y^{-1+\varepsilon},$$

which follows from (1) by partial summation.

Now our Theorem follows from (3)-(6).

References

[1] Cundian Yang and Chao Li, Asymptotion Formulae of Smarandache-Type Multiplicative Functions, Hexis, 2004, pp. 139-142.

[2] A. Ivić, The Riemann zeta-function, Wiley, New York, 1985.