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Min-Max Dom-Saturation Number of a Tree

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Abstract: Let $G = (V,E)$ be a graph and let $v \in V$. Let $\gamma^{min}(v,G)$ denote the minimum cardinality of a minimal dominating set of G containing v. Then $\gamma^{M,m}(G) = max\{\gamma^{min}(v,G) : v \in V(G)\}$ is called the min-max dom-saturation number of G. In this paper we present a dynamic programming algorithm for determining the min-max dom-saturation number of a tree.

Key Words: Domination, Smarandachely k-dominating set, min-max dom-saturation number.

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§1. Introduction

By a graph $G = (V,E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [6].

One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching. An excellent treatment of fundamentals of domination in graphs is given in the book by Haynes et al.[7]. Surveys of several advanced topics in domination are given in the book edited by Haynes et al.[8].

Let $G = (V,E)$ be a graph. A subset S of V is said to be a Smarandachely k-dominating set in G if every vertex in $V - S$ is adjacent to at least k vertices in S. When $k = 1$, the set S is simply called a *dominating set*. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set of G. The domination number $\gamma(G)$ is the minimum cardinality taken over all minimal dominating sets in G.

Let S be a subset of vertices of a graph G and let $u \in S$. A vertex v is called a private neighbor of u with respect to S if $N[v] \cap S = \{u\}$. A dominating set D of G is a minimal dominating set if and only if every vertex in D has a private neighbor with respect to D.

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In a graph G any vertex of degree 1 is called a leaf and the unique vertex which is adjacent to a leaf is called a support vertex.

Acharya [1] introduced the concept of dom-saturation number $ds(G)$ of a graph, which is defined to be the least positive integer k such that every vertex of G lies in a dominating set of cardinality k. Arumugam and Kala [2] observed that for any graph G, $ds(G) = \gamma(G)$ or $\gamma(G)+1$ and obtained several results on $ds(G)$. Motivated by this concept Arumugam and Subramanian [3] introduced the concept of independence saturation number of a graph and Arumugam et al. [4] introduced the concept of irredundance saturation number of a graph. In [5] we have generalized the concept of min-max and max-min graph saturation parameters for any graph theoretic property P which may be hereditary or super hereditary in the following.

Definition 1.1 *The min-max dom-saturation number $\gamma^{M,m}(G)$ is defined as follows. For any $v \in V(G)$, let $\gamma^{min}(v, G) = min\{|S| : S$ is a minimal dominating set of G and $v \in S\}$ and let $\gamma^{M,m}(G) = max\{\gamma^{min}(v, G) : v \in V(G)\}$.*

Thus $\gamma^{M,m}(G)$ is the largest positive integer k, with the property that every vertex of G lies in a minimal dominating set of cardinality at least k.

Since the decision problem corresponding to the domination number $\gamma(G)$ is NP-complete, it follows that the decision problem corresponding to $\gamma^{M,m}(G)$ is also NP-complete. Hence developing polynomial time algorithms for determining $\gamma^{M,m}(G)$ for special classes of graphs is an interesting problem.

In this paper we present a dynamic programming algorithm for determining the min-max dom-saturation number of a tree.

§2. Main Results

Let T be a tree rooted at v. For any vertex $u \in V(T)$, let T_u be the subtree of T rooted at u. Let $u_1, \ldots, u_k$ be the children of u in T_u and let $T_i = T_{u_i}$. For any dominating set D of T_u, let $D_i = D \cap V(T_i)$. We now define the following six parameters.

(i) $\gamma^1(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \in D$ and u is isolated in $\langle D\rangle\}$.

(ii) $\gamma^2(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \in D$, u is not isolated in $\langle D\rangle$ and u has a child as its private neighbor$\}$.

(iii) $\gamma^3(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \notin D$ and u is a private neighbor of its child$\}$.

(iv) $\gamma^4(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u - u$ and $u_i \notin D, 1 \leq i \leq k\}$.

(v) $\gamma^5(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \notin D$ and at least two of its children are in $D\}$.

(vi) $\gamma^{00}(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u - u\}$.

Observation 2.1 If the subtree T_u is a star or if every child of u is a support vertex, then $\gamma^2(T,u)$ is not defined. Also if the vertex u has two leaves as its children then $\gamma^3(T,u)$ is not defined. If u is a support vertex of T_u, then $\gamma^4(T,u)$ is not defined and if the number of children of u is less than two then $\gamma^5(T,u)$ is not defined.

Lemma 2.1 $\gamma^1(T,u) = 1 + \sum\limits_{i=1}^{k} min\{\gamma^4(T_i,u_i), \gamma^5(T_i,u_i), \gamma^{00}(T_i,u_i)\}$.

Proof Let D be a minimal dominating set of T_u, $u \in D, u$ is isolated in $\langle D\rangle$ and $|D| = \gamma^1(T,u)$. Hence $u_i \notin D_i, 1 \le i \le k$. If no children of u_i is in D_i, then $|D_i| \ge \gamma^{00}(T_i,u_i)$. If exactly one child of u_i is in D_i, then $|D_i| \ge \gamma^4(T_i,u_i)$. Otherwise $|D_i| \ge \gamma^5(T_i,u_i)$. Thus $|D_i| \ge min\{\gamma^4(T_i,u_i), \gamma^5(T_i,u_i), \gamma^{00}(T_i,u_i)\}$. Hence $|D| \ge 1+\sum\limits_{i=1}^{k} min\{\gamma^4(T_i,u_i), \gamma^5(T_i,u_i), \gamma^{00}(T_i,u_i)\}$. We get the equality. □

The reverse inequality follows from the observation that any minimal dominating set D of T_u having u as an isolated vertex in $\langle D\rangle$ is of the form $D = \left(\bigcup\limits_{i=1}^{k} D_i\right) \cup \{u\}$ where D_i is a minimal dominating set of T_i not containing $u_i, 1 \le i \le k$.

Lemma 2.2 *Suppose the subtree T_u of T rooted at u is neither a star nor every child of u is a support vertex. Then* $\gamma^2(T,u) = 1 + \min\limits_{i,j} \{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + \gamma^4(T_j,u_j) + \sum\limits_{r\ne i,j} min\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^4(T_r,u_r), \gamma^5(T_r,u_r), \gamma^{00}(T_r,u_r)\}\}$ *where the minimum is taken over all i,j such that u_i is not a leaf of T_u and u_j is not a support vertex of T_u.*

Proof Let D be a minimal dominating set of $T_u, u \in D$, u is not isolated in $\langle D\rangle$ and u has one of its children as its private neighbor and $|D| = \gamma^2(T,u)$. Without loss of generality we assume that $u_i \in D$ and u_j is the private neighbor of u with respect to D. Since D is a minimal dominating set it follows that u_i is not a leaf of T_u and u_j is not a support vertex of T_u. Since $u_i \in D, |D_i| \ge min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\}$. Also u_j and all its children are not in D_j, we have $|D_j| \ge \gamma^4(T_j,u_j)$. For $r \ne i,j$,

$$|D_r| \ge min\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^4(T_r,u_r), \gamma^5(T_r,u_r), \gamma^{00}(T_r,u_r)\}.$$

Hence

$$\begin{aligned} |D| \quad \ge \quad & 1 + \min_{i,j}\{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + \gamma^4(T_j,u_j) \\ & + \sum_{r\ne i,j} min\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^4(T_r,u_r), \gamma^5(T_r,u_r), \gamma^{00}(T_r,u_r)\}\}, \end{aligned}$$

where the minimum is taken over all i,j such that u_i is not a leaf of T_u and u_j is not a support vertex of T_u.

The reverse inequality is obvious. □

Lemma 2.3 *Let D be a minimal dominating set of T_u such that $u \notin D$. If a child of u, say u_1 is a leaf, then* $\gamma^3(T,u) = 1 + \sum\limits_{i=2}^{k} min\{\gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$. *If no child of u is a leaf, then* $\gamma^3(T,u) = \min\limits_{1\le i\le k}\{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + \sum\limits_{j\ne i} min\{\gamma^3(T_j,u_j), \gamma^5(T_j,u_j)\}\}$.

Proof Let D be a minimal dominating set of T_u such that $u \notin D$, u is a private neighbor of a child and $|D| = \gamma^3(T,u)$.

Case 1. Exactly one child, say u_1, of u is a leaf.

Then $u_1 \in D$ and $u_i \notin D$ for all $i > 1$.

Hence $\gamma^3(T,u) \geq 1 + \sum\limits_{i=2}^{k} min\{\gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$.

Case 2. No child of u is a leaf.

Without loss of generality we assume that u is the private neighbor of $u_i \in D$. Then $|D_i| \geq min\{\gamma^1(T_i,u_i),\ \gamma^2(T_i,u_i)\}$. Also since u is the private neighbor of u_i, all the other children of u are not in D and hence for all $j \neq i$,

$$|D_j| \geq min\{\gamma^3(T_j,u_j), \gamma^5(T_j,u_j)\}.$$

Thus $|D| \geq \min\limits_{1\leq i\leq k}\{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + \sum\limits_{j\neq i} min\{\gamma^3(T_j,u_j), \gamma^5(T_j,u_j)\}\}$.

The reverse inequality is obvious. □

Lemma 2.4 *If u is not a support vertex of T_u, then*

$$\gamma^4(T,u) = \sum_{i=1}^{k} min\{\gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}.$$

Proof Let D be a minimal dominating set of $T_u - \{u\}$, $u_i \notin D$ and $|D| = \gamma^4(T,u)$. Let $D_i = D \cap V(T_i)$. Since $u_i \notin D_i, |D_i| \geq min\{\gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$ and hence $|D| \geq \sum\limits_{i=1}^{k} min\{\gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$. The reverse inequality is obvious. □

Lemma 2.5 *If u has more than one child, then*

$$\begin{aligned}\gamma^5(T,u) &= \min_{i,j}\{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + min\{\gamma^1(T_j,u_j), \gamma^2(T_j,u_j)\} \\ &\quad + \min_{r\neq i,j}\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^3(T_r,u_r), \gamma^5(T_r,u_r)\}\}.\end{aligned}$$

Proof Let D be a minimal dominating set of T_u such that at least two children of u, say u_i and u_j are in D and $|D| = \gamma^5(T,u)$. Since $u_i, u_j \in D$, $|D_i| \geq min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\}$ and $|D_j| \geq min\{\gamma^1(T_j,u_j), \gamma^2(T_j,u_j)\}$. For any $r \neq i,j, u_r$ may or may not be in D. Hence

$$|D_r| \geq min\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^3(T_r,u_r), \gamma^5(T_r,u_r)\}.$$

Thus

$$\begin{aligned}|D| &\geq \min_{i,j}\{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + min\{\gamma^1(T_j,u_j), \gamma^2(T_j,u_j)\} \\ &\quad + \min_{r\neq i,j}\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^3(T_r,u_r), \gamma^5(T_r,u_r)\}\}.\end{aligned}$$

The reverse inequality is obvious. □

Lemma 2.6 $\gamma^{00}(T,u) = \sum_{i=1}^{k} min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i), \gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$.

Proof Let D be a minimal dominating set of $T_u - u$ such that $|D| = \gamma^{00}(T,u)$. Obviously $|D_i| \geq min\{\gamma^1(T_i,u_i),\ \gamma^2(T_i,u_i), \gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$. Thus

$$|D| \geq \sum_{i=1}^{k} min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i), \gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}.$$

The reverse inequality is obvious. □

Lemma 2.7 $\gamma^{min}(v,T) = min\{\gamma^1(T,v), \gamma^2(T,v)\}$.

Proof Let D be a minimal dominating set of T such that $v \in D$ and $|D| = \gamma^{min}(v,T)$. Since v is either isolated or nonisolated in $\langle D\rangle$, the result follows. □

Based on the above lemmas we have the following dynamic programming algorithm for determining $\gamma^{min}(v,T)$ for trees.

ALGORITHM TO FIND $\gamma^{min}(v,T)$

INPUT: A tree T rooted at v_1, with a BFS ordering of its vertices $\{v_1, v_2, \ldots, v_n\}$.
OUTPUT: Minimum cardinality of a minimal dominating set of T containing v_1.

Step 1. `INITIALIZATION`

for $i = 1$ to n **do**

$\gamma^1(v_i) = 1; \gamma^2(v_i) = \infty; \gamma^3(v_i) = \infty,$

$\gamma^4(v_i) = \infty; \gamma^5(v_i) = \infty; \gamma^{00}(v_i) = 0.$

end for;

Step 2. `COMPUTATION`

for $i = n$ to 1 **do**

Step 2.1: Let $u_{i1}, u_{i2}, \ldots, u_{il}$ be the children of v_i

Step 2.2: `CALCULATE` $\gamma^1(v_i)$

Compute $\gamma^1(v_i) = 1 + \sum_{j=1}^{l} min\{\gamma^4(u_{ij}), \gamma^5(u_{ij}), \gamma^{00}(u_{ij})\}$.

Step 2.3: `CALCULATE` $\gamma^2(v_i)$

If there exists a child of v_i which is not a leaf and there exists a child of v_i which is not a support then compute

$$\gamma^2(v_i) = 1 + \min_{j,k}\{min\{\gamma^1(u_{ij}), \gamma^2(u_{ij})\}+ \gamma^4(u_{ik}) + \sum_{r\neq j,k}\{\gamma^1(u_{ir}), \gamma^2(u_{ir}), \gamma^4(u_{ir}), \gamma^5(u_{ir}), \gamma^{00}(u_{ir})\}.$$

where the minimum is taken over all $j, k, j \neq k$ such that u_{ik} is not a support vertex and u_{ij} is not a leaf.

Step 2.4: `CALCULATE` $\gamma^3(v_i)$

If v_i has exactly one child which is a leaf, say u_1, then compute $\gamma^3(v_i) = 1 + \sum_{j=2}^{l} min\{\gamma^3(u_{ij}), \gamma^5(u_{ij})\}$

otherwise

$\gamma^3(v_i) = \min_{1\le j\le l}\{min\{\gamma^1(u_{ij}), \gamma^2(u_{ij})\} + \sum_{k\neq j}\{\gamma^3(u_{ik}), \gamma^5(u_{ik})\}\}$.

Step 2.5: `CALCULATE` $\gamma^4(v_i)$

If v_i is not a support vertex then compute

$\gamma^4(v_i) = \sum_{j=1}^{l} min\{\gamma^3(u_{ij}), \gamma^5(u_{ij})\}$

Step 2.6: `CALCULATE` $\gamma^5(v_i)$

If v_i has more than one child then compute

$$\gamma^5(v_i) = \min_{j\neq k}\{\gamma^1(u_{ij}), \gamma^2(u_{ij})\} + min\{\gamma^1(u_{ik}), \gamma^2(u_{ik})\} + \min_{r\neq j,k}\{\gamma^1(u_{ir}), \gamma^2(u_{ir}), \gamma^3(u_{ir}), \gamma^5(u_{ir})\}$$

Step 2.7: `CALCULATE` $\gamma^{00}(v_i)$

Compute $\gamma^{00}(v_i) = \sum_{j=1}^{l}\{\gamma^1(u_{ij}), \gamma^2(u_{ij}), \gamma^3(u_{ij}), \gamma^5(u_{ij})\}$

end for;

Step 3. Compute $\gamma^{min}(v_1, T) = min\{\gamma^1(v_1), \gamma^2(v_1)\}$.

Observation 2.2 Using the above algorithm for any given vertex v of T the parameter $\gamma^{min}(v, T)$ can be computed. Applying the above algorithm for each vertex v we compute $\gamma^{min}(v, T)$ for all $v \in V$ and $\gamma^{M,m}(T) = max\{\gamma^{min}(v, T) : v \in V(T)\}$ can be computed.

Example 2.1 A tree rooted at the vertex 1 and the table showing the computations of the above algorithm are given below.

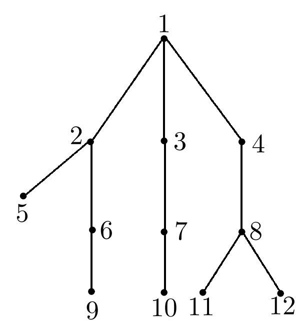

Figure 1

	γ^1	γ^2	γ^3	γ^4	γ^5	γ^{00}
12	1	∞	∞	∞	∞	0
11	1	∞	∞	∞	∞	0
10	1	∞	∞	∞	∞	0
9	1	∞	∞	∞	∞	0

	γ^1	γ^2	γ^3	γ^4	γ^5	γ^{00}
8	1	∞	∞	∞	2	2
7	1	∞	1	∞	∞	1
6	1	∞	1	∞	∞	1
5	1	∞	∞	0	∞	0
4	3	∞	1	2	∞	1
3	2	∞	1	1	∞	1
2	2	2	2	∞	2	2
1	5	5	4	4	5	4

Hence $\gamma^{min}(1,T) = min(\gamma^1(T,1), \gamma^2(T,1)) = 5$.

Repeated application of the algorithm gives $\gamma^{min}(2,T) = 4$, $\gamma^{min}(3,T) = 5$, $\gamma^{min}(4,T) = 5$, $\gamma^{min}(5,T) = 5$, $\gamma^{min}(6,T) = 4$, $\gamma^{min}(7,T) = 4$, $\gamma^{min}(8,T) = 4$, $\gamma^{min}(9,T) = 4$, $\gamma^{min}(10,T) = 5$, $\gamma^{min}(11,T) = 6$, $\gamma^{min}(12,T) = 6$. Hence $\gamma^{M,m}(T) = max\{\gamma^{min}(i,T) : 1 \leq i \leq 12\} = 6$.

§3. Conclusion

Courcelle has proved that if a graph property can be expressed in extended monadic second order logic (EMSO), then for every fixed $w \geq 1$, there is a linear-time algorithm for testing this property on graphs having treewidth at most w. The property of a subset S of V being a minimal dominating set can be expressed in EMSO and hence for families of graphs with bounded treewidth, a linear time algorithm can be developed for computing $\gamma^{min}(v,G)$ for any given vertex v. Hence developing such algorithm for specific families of graphs of bounded treewidth is an interesting problem for further research.

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