Scientia Magna
Vol. 2 (2006), No. 2, 77-81

On the K-power free number sequence

Qing Tian

Department of Mathematics, Northwest University
Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is to study the distribution properties of the k-power free numbers, and give an interesting asymptotic formula for it.

Keywords k-power free number, mean value, asymptotic formula

§1. Introduction

A natural number n is called a k-power free number if it can not be divided by any p^k, where p is a prime. One can obtain all k-power free number by the following method: From the set of natural numbers (except 0 and 1).

$-$take off all multiples of 2^k, ($i.e$ $2^k, 2^{k+1}, 2^{k+2}\ldots$).

$-$take off all multiples of 3^k.

$-$take off all multiples of 5^k.

... and so on (take off all multiples of all k-power primes).

When $k = 3$, the $k-$power free number sequence is 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17,$\cdots$. In reference [1], Professor F. Smarandache asked us to study the properties of the $k-$power free number sequence. About this problem, Zhang Tianping had given an asymptotic formula in reference [3].That is, he proved that

$$\sum_{\substack{n\le x\\ n\in B}}\omega^2(n)=\frac{x(\ln\ln x)^2}{\zeta(k)}+O(x(\ln\ln x)),$$

where $\omega(n)$ denotes the number of prime divisors of n, $\zeta(k)$ is the Riemann zeta-function.

This paper as a note of [3], we use the analytic method to obtain a more accurate asymptotic formula for it. That is, we shall prove the following:

Theorem. Let k be a positive integer with $k\ge 2$, B denotes the set of all k-power free number. Then we have the asymptotic formula

$$\sum_{\substack{n\le x\\ n\in B}}\omega^2(n)=\frac{1}{\zeta(k)}\left(x(\ln\ln x)^2+C_1x\ln\ln x+C_2x\right)+O\left(\frac{x\ln\ln x}{\ln x}\right),$$

where $\zeta(k)$ is the Riemann zeta$-$function, C_1 and C_2 are both constants.

§2. Some Lemmas

To complete the proof of the theorem, we need following several Lemmas.

Lemma 1. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x}\omega(n) = x\ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

$$\sum_{n\leq x}\omega^2(n) = x(\ln\ln x)^2 + ax\ln\ln x + bx + O\left(\frac{x\ln\ln x}{\ln x}\right),$$

where $A = \gamma + \sum_p \left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$, a and b are two computable constants.

Proof. See reference [2].

Lemma 2. Let $\mu(n)$ be the Möbius function, then for any real number $x \geq 2$, we have the following identity

$$\sum_{n=1}^{\infty}\frac{\mu(n)\omega(n)}{n^s} = -\frac{1}{\zeta(s)}\sum_p \frac{1}{p^s-1}.$$

Proof. See reference [3].

Lemma 3. Let $k \geq 2$ be a fixed integer, then for any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{d^k m\leq x}\omega^2(m)\mu(d) = \frac{1}{\zeta(k)}\left(x(\ln\ln x)^2 + ax\ln\ln x + bx\right) + O\left(\frac{x\ln\ln x}{\ln x}\right).$$

Proof. From Lemma 1 we have

$$\begin{aligned}
&\sum_{d^k m\leq x}\omega^2(m)\mu(d) = \sum_{d\leq x^{\frac{1}{k}}}\mu(d)\sum_{m\leq \frac{x}{d^k}}\omega^2(m)\\
=&\sum_{d\leq x^{\frac{1}{k}}}\mu(d)\left(\frac{x}{d^k}(\ln\ln\frac{x}{d^k})^2 + a\frac{x}{d^k}\ln\ln\frac{x}{d^k} + b\frac{x}{d^k} + O\left(\frac{\frac{x}{d^k}\ln\ln\frac{x}{d^k}}{\ln\frac{x}{d^k}}\right)\right)\\
=&\ x\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\left(\ln\ln x + \ln\left(1-\frac{k\ln d}{\ln x}\right)\right)^2 + ax\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\ln\frac{x}{d^k}\\
&+bx\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k} + O\left(\frac{x\ln\ln x}{\ln x}\right). \qquad (1)
\end{aligned}$$

The first term on the right hand side of (1) is

$$\begin{aligned}
&x\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\left(\ln\ln x + \ln\left(1-\frac{k\ln d}{\ln x}\right)\right)^2\\
=&\ x(\ln\ln x)^2\sum_{d\leq x^{\frac{1}{k}}}\frac{\mu(d)}{d^k} + O\left(x\sum_{d\leq x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\cdot\frac{k\ln d}{\ln x}\cdot\ln\ln x\right) + O\left(x\sum_{d\leq x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\cdot\frac{\ln^2 d}{\ln^2 x}\right)\\
=&\ \frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right) + O\left(\frac{x\ln\ln x}{\ln x}\right) + O\left(\frac{x}{\ln^2 x}\right)\\
=&\ \frac{x(\ln\ln x)^2}{\zeta(k)} + O\left(\frac{x\ln\ln x}{\ln x}\right).
\end{aligned}$$

The second term on the right hand side of (1) is

$$ax\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\ln\frac{x}{d^k}=ax\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\ln x+ax\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}\ln\left(1-\frac{k\ln d}{\ln x}\right)$$

$$=\frac{ax\ln\ln x}{\zeta(k)}+O\left(x\sum_{d>x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\ln\ln x\right)+O\left(x\sum_{d\le x^{\frac{1}{k}}}\frac{|\mu(d)|}{d^k}\frac{\ln d}{\ln x}\right)$$

$$=\frac{ax\ln\ln x}{\zeta(k)}+O\left(\frac{x}{\ln x}\right).$$

The third term on the right hand side of (1) is

$$bx\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^k}=bx\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k}+O\left(bx\sum_{d>x^{\frac{1}{k}}}\frac{1}{d^k}\right)=\frac{bx}{\zeta(k)}+O\left(x^{\frac{1}{k}}\right).$$

From the calculations above we get the asymptotic formula

$$\sum_{d^km\le x}\omega^2(n)\mu(d)=\frac{1}{\zeta(k)}\left(x(\ln\ln x)^2+ax\ln\ln x+bx\right)+O\left(\frac{x\ln\ln x}{\ln x}\right).$$

This proves Lemma 3.

Lemma 4. Let any real number $x\ge 2$, we have the asymptotic formula

$$\sum_{d^km\le x}\omega^2(d)\mu(d)=A_1x+O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right),$$

where $A_1=\sum\limits_{d=1}^{\infty}\frac{\omega^2(d)\mu(d)}{d^k}$ is a calculable constant.

Proof. Note that the series $\sum_{d^km\le x}\omega^2(d)\mu(d)$ is convergent, so we have

$$\sum_{d^km\le x}\omega^2(d)\mu(d)=\sum_{d\le x^{\frac{1}{k}}}\omega^2(d)\mu(d)\sum_{m\le\frac{x}{d^k}}1=\sum_{d\le x^{\frac{1}{k}}}\omega^2(d)\mu(d)\left[\frac{x}{d^k}\right]$$

$$=x\sum_{d=1}^{\infty}\frac{\omega^2(d)\mu(d)}{d^k}+O\left(x\sum_{d>x^{\frac{1}{k}}}\frac{\omega^2(d)|\mu(d)|}{d^k}\right)+O\left(\sum_{d\le x^{\frac{1}{k}}}\omega^2(d)|\mu(d)|\right)$$

$$=A_1x+O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right)+O\left(x^{\frac{1}{k}}(\ln\ln x^{\frac{1}{k}})^2\right)$$

$$=A_1x+O\left(x^{\frac{1}{k}}(\ln\ln x)^2\right).$$

This proves Lemma 4.

Lemma 5. For any real number $x\ge 2$, we have the asymptotic formula

$$\sum_{d^km\le x}\omega^2((d,m))\mu(d)=A_2x+O\left(x^{\frac{1}{k}}\right),$$

where $A_2=\dfrac{\mu(d)\sum\limits_{u|d}\frac{\omega^2(u)}{u}\sum\limits_{s|\frac{d}{u}}\frac{\mu(s)}{s}}{d^k}$ is a calculable constant.

Proof. Assume that (u,v) is the greatest common divisor of u and v, then we have

$$\sum_{d^k m\le x}\omega^2((d,m))\mu(d)=\sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{\substack{m\le\frac{x}{d^k}\\ u|m,(\frac{m}{u},\frac{d}{u})=1}}\omega^2(u)$$

$$\begin{aligned}
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{\substack{m\le\frac{x}{d^k}\\ u|m}}\omega^2(u)\left[\frac{1}{(\frac{m}{u},\frac{d}{u})}\right]\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{\substack{m\le\frac{x}{d^k}\\ u|m}}\omega^2(u)\sum_{\substack{s|\frac{m}{u}\\ s|\frac{d}{u}}}\mu(s)\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\sum_{q\le\frac{x}{d^k su}}\omega^2(u)\sum_{s|\frac{d}{u}}\mu(s)\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\omega^2(u)\sum_{s|\frac{d}{u}}\mu(s)\left[\frac{x}{d^k us}\right]\\
&= \sum_{d\le x^{\frac{1}{k}}}\mu(d)\sum_{u|d}\omega^2(u)\sum_{s|\frac{d}{u}}\mu(s)\left(\frac{x}{d^k us}+O(1)\right)\\
&= x\sum_{d=1}^{\infty}\frac{\mu(d)\sum\limits_{u|d}\frac{\omega^2(u)}{u}\sum\limits_{s|\frac{d}{u}}\frac{\mu(s)}{s}}{d^k}+O\left(x\sum_{d>x^{\frac{1}{k}}}\frac{|\mu(d)|\sum\limits_{u|d}\frac{\omega^2(u)}{u}\sum\limits_{s|\frac{d}{u}}\frac{|\mu(s)|}{s}}{d^k}\right)\\
&\quad +O\left(\sum_{d\le x^{\frac{1}{k}}}|\mu(d)|\sum_{u|d}\omega^2(u)\sum_{s|\frac{d}{u}}|\mu(s)|\right)\\
&= A_2x+O\left(x^{\frac{1}{k}+\varepsilon}\right),
\end{aligned}$$

where ε is any positive number. This proves Lemma 5.

Same as the method used in the above Lemmas, we can easily get the following three asymptotic formulas:

Lemma 6. For any real number $x\ge 2$, we have

$$\sum_{d^k m\le x}\omega(d)\omega((d,m))\mu(d)=A_3x+O\left(x^{\frac{1}{k}}\ln\ln x\right),$$

$$\sum_{d^k m\le x}\omega(m)\omega((d,m))\mu(d)=A_4x\ln\ln x+O\left(\frac{x\ln\ln x}{\ln x}\right),$$

$$\sum_{d^k m\le x}\omega(d)\omega(m)\mu(d)=C\left(x\ln\ln x+Ax\right)+O\left(\frac{x\ln\ln x}{\ln x}\right),$$

where $C=-\frac{1}{\zeta(k)}\sum_p\frac{1}{p^k-1}$, A_3 and A_4 are constants.

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. From Lemmas above, we have

$$\begin{aligned}
&\sum_{\substack{n\le x\\ n\in B}}\omega^2(n)=\sum_{n\le x}\omega^2(n)\sum_{d^k|n}\mu(d)=\sum_{d^km\le x}\omega^2(d^km)\mu(d)\\
=&\sum_{d^km\le x}(\omega(d)+\omega(m)-\omega((d,m)))^2\mu(d)\\
=&\sum_{d^km\le x}\omega^2(m)\mu(d)+\sum_{d^km\le x}\omega^2(d)\mu(d)+\sum_{d^km\le x}\omega^2((d,m))\mu(d)\\
&\quad+2\left(\sum_{d^km\le x}\omega(d)\omega(m)\mu(d)\right)-2\left(\sum_{d^km\le x}\omega(d)\omega((d,m))\mu(d)\right)\\
&\quad-2\left(\sum_{d^km\le x}\omega(m)\omega((d,m))\mu(d)\right)\\
=&\frac{1}{\zeta(k)}\left(x(\ln\ln x)^2+C_1x\ln\ln x+C_2x\right)+O\left(\frac{x\ln\ln x}{\ln x}\right).
\end{aligned}$$

This completes the proof of Theorem.

References

[1] F.Smarandache, Only problems, Not solutions, Xiquan Publishing House, Chicago, 1993, 27

[2] G. H. Hardy and S. Ramanujan, The normal number of prime factors of a number n, Quart. J. Math, **48**(1917), 76-92

[3] Zhang Tianping, On the k-power free number sequence, Smarandache Notions Journal, **14**(2004), 62-65