

ON A FUNCTION IN THE NUMBERS THEORY

by

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Abstract. In the present paper we study some series concerning the following function of the Numbers Theory [1]: "S : $\mathbb{N} \rightarrow \mathbb{N}$ such that S(n) is the smallest k with property that k! is divisible by n".

1. **Introduction.** The following functions in Numbers Theory are well - known : the function $\mu(n)$ of M \hat{o} bius, the function $\xi(s)$ of Riemann ($\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s = \sigma + it \in \mathbb{C}$), the function $\Lambda(n)$ of Mangoldt $\left(\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \\ 0, & \text{if } n \neq p^m \end{cases} \right)$ etc.

The purpose of this paper is to study some series concerning the following function of the Numbers Theory "[1] S : $\mathbb{N} \rightarrow \mathbb{N}$ such that S(n) is the smallest integer k with the propriety that k! is divisible by n".

We first prove the divergence of some series involving the S function, using an unitary method, and then we prove that the series $\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$ is convergent to a number $S \in (71/100, 101/100)$ and we study some applications of this series in the Numbers Theory .

Then we prove that series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a real numbers $s \in (0.717, 1.253)$ and that the sum of the remarkable series $\sum_{n \geq 2} \frac{S(n)}{n!}$ is a irrational number.

2. The main results

Proposition 1. If $(x_n)_{n \geq 1}$ is strict increasing sequence of natural numbers, then the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \quad (1)$$

is divergent.

Proof. We consider the function $f: [x_n, x_{n+1}] \rightarrow \mathbb{R}$, defined by $f(x) = \ln \ln x$ is meets the conditions of the Lagrange's theorem of finite increases. Therefore there is $c_n \in (x_n, x_{n+1})$ such that :

$$\ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n} (x_n - x_{n+1}). \quad (2)$$

Because $x_n < c_n < x_{n+1}$, we have :

$$\frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, (\forall) n \in \mathbb{N}, \quad (3)$$

if $x_n \neq 1$.

We know that for each $n \in \mathbb{N} \setminus \{1\}$, $\frac{S(n)}{n} \leq 1$, i.e.

$$0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n}, \quad (4)$$

from where it results that $\lim_{n \rightarrow \infty} \frac{S(n)}{n \ln n} = 0$. Hence there is $k > 0$ such that $\frac{S(n)}{n \ln n} < k$, i.e., $n \ln n > \frac{S(n)}{k}$ for any $n \in \mathbb{N}^*$, so

$$\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}. \quad (5)$$

Introducing (5) in (3) we obtain :

$$\ln \ln x_{n+1} - \ln \ln x_n < k \frac{x_{n+1} - x_n}{S(x_n)}, (\forall) n \in \mathbb{N}^* \setminus \{1\}. \quad (6)$$

Summing up after n it results :

$$\sum_{n=1}^m \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k} (\ln \ln x_{m+1} - \ln \ln x_1).$$

Because $\lim_{m \rightarrow \infty} x_m = \infty$ we have $\lim_{m \rightarrow \infty} \ln \ln x_m = \infty$, i.e., the series :

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$$

is divergent. The Proposition 1 is proved.

Proposition 2. Series $\sum_{n=2}^{\infty} \frac{1}{S(n)}$ is divergent.

Proof. We use Proposition 1 for $x_n = n$.

Remarks.

- 1) If x_n is the n -th prime number, then the series $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$ is divergent.
- 2) If the sequence $(x_n)_{n \geq 1}$ forms an arithmetical progression of natural numbers, then the series $\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$ is divergent.
- 3) The series $\sum_{n=1}^{\infty} \frac{1}{S(2n+1)}$, $\sum_{n=1}^{\infty} \frac{1}{S(4n+1)}$ etc., are all divergent.

In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent forms are divergent.

Proposition 3. The series

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$$

is convergent to a number $s \in (71/100, 101/100)$.

Proof. From the definition it results $S(n) \leq n!$, $(\forall) n \in \mathbb{N}^* \setminus \{1\}$, so $\frac{1}{S(n)} \geq \frac{1}{n!}$.

Summing up, beginning with $n=2$ we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

The product $S(2) S(3) \dots S(n)$ is greater than the product of prime numbers from the set $\{1, 2, \dots, n\}$, because $S(p)=p$, for $p=$ prime number. Therefore :

$$\frac{1}{\prod_{i=2}^n S(i)} < \frac{1}{\prod_{i=2}^n p_i}, \quad (7)$$

where p_k is the biggest number smaller or equal to n .

There are the inequalities :

$$\begin{aligned}
S &= \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2)S(3)} + \frac{1}{S(2)S(3)S(4)} + \dots + \\
&\quad + \frac{1}{S(2)S(3)\dots S(k)} + \dots < \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} + \frac{2}{2 \cdot 3 \cdot 5 \cdot 7} + \\
&\quad + \frac{2}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} + \dots + \frac{p_{k+1} - p_k}{p_1 p_2 \dots p_k} + \dots
\end{aligned} \tag{8}$$

Using the inequality $p_1 p_2 \dots p_k > p_{k+1}^3$, $(\forall) k \geq 5$ [5], we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6^2} + \frac{1}{p_7^2} + \dots + \frac{1}{p_{k+1}^2} + \dots \tag{9}$$

We symbolise by $P = \frac{1}{p_6^2} + \frac{1}{p_7^2} + \dots$ and observe that $P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \dots$

It results :

$$P < \frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{12^2} \right),$$

where

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \text{ (EULER).}$$

Introducing in (9) we obtain :

$$S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{1}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \dots - \frac{1}{12^2}.$$

Estimating with an approximation of an order not more than $\frac{1}{10^2}$, we find :

$$0,71 < \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} < 0,79. \tag{10}$$

The proposition 3 is proved.

Remark. Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right ranging :

$$\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)} < 0,97. \tag{11}$$

Proposition 4. Let α be a fixed real number, $\alpha \geq 1$. Then the series $\sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\dots S(n)}$

is convergent.

Proof. Be $(x_k)_{k \geq 1}$ the sequence of prime numbers. We can write :

$$\frac{2^\alpha}{S(2)} = \frac{2^\alpha}{2} = 2^{\alpha-1}$$

$$\frac{3^\alpha}{S(2)S(3)} = \frac{3^\alpha}{p_1 p_2}$$

$$\frac{4^\alpha}{S(2)S(3)S(4)} < \frac{4^\alpha}{p_1 p_2} < \frac{p_3^\alpha}{p_1 p_2}$$

$$\frac{5^\alpha}{S(2)S(3)S(4)S(5)} < \frac{5^\alpha}{p_1 p_2 p_3} < \frac{p_4^\alpha}{p_1 p_2 p_3}$$

$$\frac{6^\alpha}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^\alpha}{p_1 p_2 p_3} < \frac{p_4^\alpha}{p_1 p_2 p_3}$$

$$\frac{n^\alpha}{S(2)S(3)\dots S(n)} < \frac{n^\alpha}{p_1 p_2 \dots p_k} < \frac{p_{k+1}^\alpha}{p_1 p_2 \dots p_k},$$

where $p_i \leq n, i \in \{1, \dots, k\}, p_{k+1} > n$.

Therefore

$$\sum_{n=2}^{\infty} \frac{n^{\alpha}}{S(2)S(3)\dots S(n)} < 2^{\alpha-1} + \sum_{n=2}^{\infty} \frac{(p_{k+1} - p_k) \cdot p_{k+1}^{\alpha}}{S(2)S(3)\dots S(n)} < 2^{\alpha-1} + \sum_{n=2}^{\infty} \frac{p_{k+1}^{\alpha}}{p_1 p_2 \dots p_k}.$$

Then it exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ we have :

$$p_1 p_2 \dots p_k > p_{k+1}^{\alpha+3}.$$

Therefore

$$\sum_{n=2}^{\infty} \frac{n^{\alpha}}{S(2)S(3)\dots S(n)} < 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^{\alpha+1}}{p_1 p_2 \dots p_k} + \sum_{k \geq k_0} \frac{1}{p_{k+1}^2}.$$

Because the series $\sum_{k \geq k_0} \frac{1}{p_{k+1}^2}$ is convergent it results that the given series is convergent

too .

Consequence 1. It exists $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$ we have $S(2)S(3)\dots S(n) > n^{\alpha}$.

Proof. Because $\lim_{n \rightarrow \infty} \frac{n^{\alpha}}{S(2)S(3)\dots S(n)} = 0$, there is $n_0 \in \mathbb{N}$ so that $\frac{n^{\alpha}}{S(2)S(3)\dots S(n)} < 1$

for each $n \geq n_0$.

Consequence 2. It exists $n_0 \in \mathbb{N}$ so that :

$$S(2) + S(3) + \dots + S(n) > (n-1)n^{\frac{\alpha}{n-1}} \text{ for each } n \geq n_0 .$$

Proof. We apply the inequality of averages to the numbers $S(2), S(3), \dots, S(n)$:

$$S(2) + S(3) + \dots + S(n) > (n-1)^{n-1} \sqrt[n-1]{S(2)S(3)\dots S(n)} > (n-1)n^{\frac{\alpha}{n-1}}, \forall n \geq n_0.$$

We can write it as it follows :

$$\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{3!} + \dots = \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \dots = \sum_{n=2}^{\infty} \frac{a(n)}{n!}, \text{ where } a(n) \text{ is the}$$

number of solutions for the equation $S(x) = n, n \in \mathbb{N}, n \geq 2$.

It results from the equality $S(x)=n$ that x is a divisor of $n!$, so $a(n)$ is smaller than $d(n!)$.

So, $a(n) < d(n!)$.

Lemma 1. We have the inequality :

$$d(n) \leq n-2, \text{ for each } n \in \mathbb{N}, n \geq 7. \tag{12}$$

Proof. Be $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ with p_1, p_2, \dots, p_k prime numbers, and $a_i \geq 1$ for each $i \in \{1, 2, \dots, k\}$. We consider the function $f: [1, \infty) \rightarrow \mathbb{R}, f(x) = a^x - x - 2, a \geq 2$, fixed. It is derivable on $[1, \infty)$ and $f'(x) = a^x \ln a - 1$. Because $a \geq 2$, and $x \geq 1$ it results that $a^x \geq 2$, so $a^x \ln a \geq 2 \ln a = \ln a^2 \geq \ln 4 > \ln e = 1$, $f'(x) > 0$ for each $x \in [1, \infty)$ and $a \geq 2$, fixed. But $f(1) = a-3$. It results that for $a \geq 3$ we have $f(x) \geq 0$ means $a^x \geq x+2$.

Particularly, for $a = p_i, i \in \{1, 2, \dots, k\}$, we obtain $p_i^{a_i} \geq a_i + 2$ for each $p_i \geq 3$.

If $n = 2^s, s \in \mathbb{N}^*,$ then $d(n) = s + 1 < 2^s - 2 = n - 2$ for $s \geq 3$.

So we can assume $k \geq 2$, i.e. $p_2 \geq 3$. The following inequalities result :

$$p_1^{a_1} \geq a_1 + 1$$

$$p_2^{a_2} \geq a_2 + 1$$

.....

$$p_k^{a_k} \geq a_k + 1,$$

equivalent with

$$p_1^{a_1} \geq a_1 + 1, p_2^{a_2} - 1 \geq a_2 + 1, \dots, p_k^{a_k} - 1 \geq a_k + 1. \tag{13}$$

By multiplying, member with member, of the inequalities (13) we obtain :

$$p_1^{a_1}(p_2^{a_2} - 1)\dots(p_k^{a_k} - 1) \geq (a_1 + 1)(a_2 + 1)\dots(a_k + 1) = d(n). \quad (14)$$

Considering the obvious inequality :

$$n - 2 \geq p_1^{a_1}(p_2^{a_2} - 1)\dots(p_k^{a_k} - 1) \quad (15)$$

and using (14) it results that :

$$n - 2 \geq d(n) \text{ for each } n \geq 7.$$

Lemma 2. $d(n!) < (n-2)!$ for each $n \in \mathbb{N}$, $n \geq 7$. (16)

Proof. We carry out an induction after n . So, for $n=7$,

$$d(7!) = d(2^4 \cdot 2^3 \cdot 5 \cdot 7) = 60 < 120 = 5!.$$

We assume that $d(n!) < (n-2)!$.

$$d((n+1)!) = d(n!(n+1)) < d(n!) d(n+1) < (n-2)!d(n+1) < (n-2)!(n-1) = (n-1)!,$$

because in according to Lemma 1, $d(n+1) < n-1$.

Proposition 5. The series $\sum_{n=2}^{\infty} \frac{1}{S(n)!}$ is convergent to a number $s \in (0.717, 1.253)$.

Proof. From Lemma 2 it results that $a(n) < (n-2)!$, so $\frac{a(n)}{n!} < \frac{1}{n(n-1)}$ for every $n \in \mathbb{N}$,

$$n \geq 7 \text{ and } \sum_{n=2}^{\infty} \frac{1}{S(n)!} = \sum_{n=2}^6 \frac{a(n)}{n!} + \sum_{n=7}^{\infty} \frac{1}{(n-1)!}.$$

$$\text{Therefore } \sum_{n=2}^{\infty} \frac{a(n)}{n!} < \frac{1}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{8}{5!} + \frac{14}{6!} + \sum_{n=7}^{\infty} \frac{1}{n^2 - n}. \quad (17)$$

$$\text{Because } \sum_{n=2}^{\infty} \frac{1}{n^2 - n} = 1 \text{ we have there is a number } s > 0, s = \sum_{n=2}^{\infty} \frac{1}{S(n)!}.$$

From (17) we obtain :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} < \frac{391}{360} + 1 - \frac{1}{2^2 - 2} - \frac{1}{3^2 - 3} - \frac{1}{4^2 - 4} + \frac{1}{5^2 - 5} + \frac{1}{6^2 - 6} = \frac{751}{360} - \frac{5}{6} = \frac{451}{360} < 1,253.$$

But, because $S(n) \leq n$ for every $n \in \mathbb{N}^*$, it results :

$$\sum_{n=2}^{\infty} \frac{1}{S(n)!} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

Consequently, for the number s we obtain the range $e-2 < s < 1,253$, i.e., $0,717 < s < 1,253$.

Because $S(n) \leq n$, it results $\sum_{n \geq 2} \frac{S(n)}{n!} \leq \sum_{n \geq 2} \frac{1}{(n-1)!}$. Therefore the series $\sum_{n \geq 2} \frac{S(n)}{n!}$ is convergent to a number f .

Proposition 6. The sum f of the series $\sum_{n \geq 2} \frac{S(n)}{n!}$ is an irrational number.

Proof. From the above results that $\lim_{n \rightarrow \infty} \sum_{i=2}^n \frac{S(i)}{i!} = 1$. Under these circumstances that $f \in \mathbb{Q}$, $f > 0$. Therefore it exists $a, b \in \mathbb{N}$, $(a, b) = 1$, so that $f = \frac{a}{b}$.

Let p be a fixed prime number, $p > b$, $p \geq 3$. Obviously, $\frac{a}{b} = \sum_{i=2}^{p-1} \frac{S(i)}{i!} + \sum_{i \geq p} \frac{S(i)}{i!}$ which leads to :

$$\frac{(p-1)!a}{b} = \sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} + \sum_{i \geq p} \frac{(p-1)!S(i)}{i!}.$$

Because $p > b$ results that $\frac{(p-1)!a}{b} \in \mathbb{N}$ and $\sum_{i=2}^{p-1} \frac{(p-1)!S(i)}{i!} \in \mathbb{N}$. Consequently we have $\sum_{i \geq p} \frac{(p-1)!S(i)}{i!} \in \mathbb{N}$ too.

Be $\alpha = \sum_{i \geq p} \frac{(p-1)!S(i)}{i!} \in \mathbb{N}$. So we have the relation

$$\alpha = \frac{(p-1)!S(p)}{p!} + \frac{(p-1)!S(p+1)}{(p+1)!} + \frac{(p-1)!S(p+2)}{(p+2)!} + \dots$$

Because p is a prime number it results $S(p)=p$.

So

$$\alpha \geq 1 + \frac{S(p+1)}{p(p+1)} + \frac{S(p+2)}{p(p+1)(p+2)!} + \dots > 1 \quad (18)$$

We know that $S(p+1) \leq p+1 (\forall i \geq 1)$, with equality only if the number $p+i$ is prime.

Consequently, we have

$$\alpha < 1 + \frac{1}{p} + \frac{1}{p(p+1)} + \frac{1}{p(p+1)(p+2)} + \dots < 1 + 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots = \frac{p}{p-1} < 2 \quad (19)$$

From the inequalities (18) and (19) results that $1 < \alpha < 2$, impossible, because $\alpha \in \mathbb{N}$.

The proposition is proved.

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