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A generalization of the Smarandache function

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Abstract For any positive integer n, we define the function $P(n)$ as the smallest prime p such that $n \mid p!$. That is, $P(n) = \min\{p : \; n \mid p!, \text{ where } p \text{ be a prime}\}$. This function is a generalization of the famous Smarandache function $S(n)$. The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of $P(n)$, and give two interesting mean value formulas for it.

Keywords The Smarandache function, generalization, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \; n \mid m!, \; n \in N\}$. For example, the first few values of $S(n)$ are: $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots\cdots$.

About the elementary properties of $S(n)$, many authors had studied it, and obtained a series results, see references [1], [2], [3], [4] and [5]. In reference [6], Jozsef Sandor introduced another arithmetical function $P(n)$ as follows: $P(n) = \min\{p : \; n \mid p!, \text{ where } p \text{ be a prime}\}$. That is, $P(n)$ denotes the smallest prime p such that $n \mid p!$. In fact function $P(n)$ is a generalization of the Smarandache function $S(n)$. Its some values are: $P(1) = 2$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, $P(5) = 5$, $P(6) = 3$, $P(7) = 7$, $P(8) = 5$, $P(9) = 7$, $P(10) = 5$, $P(11) = 11$, $\cdots$. It is easy to prove that for each prime p one has $P(p) = p$, and if n is a square-free number, then $P(n) =$ greatest prime divisor of n. If p be a prime, then the following double inequality is true:

$$2p + 1 \le P(p^2) \le 3p - 1.$$

For any positive integer n, one has (See Proposition 4 of reference [6])

$$S(n) \le P(n) \le 2S(n) - 1. \tag{1}$$

The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the function $P(n)$, and give two interesting mean value formulas it. That is, we shall prove the following conclusions:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} P(n) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

Theorem 2. For any real number $x > 1$, we also have the mean value formula

$$\sum_{n\le x} \left(P(n) - \overline{P}(n)\right)^2 = \frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot\frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\overline{P}(n)$ denotes the largest prime divisor of n, and $\zeta(s)$ is the Riemann zeta-function.

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. For any real number $x > 1$, we divide all positive integers in the interval $[1,\ x]$ into two subsets A and B, where A denotes the set of all integers $n \in [1,\ x]$ such that there exists a prime p with $p|n$ and $p > \sqrt{n}$. And B denotes the set involving all integers $n \in [1,\ x]$ with $n \notin A$. From the definition and properties of $P(n)$ we have

$$\sum_{n\in A} P(n) = \sum_{\substack{n\le x\\ p|n,\ \sqrt{n}<p}} P(n) = \sum_{\substack{pn\le x\\ n<p}} P(pn) = \sum_{\substack{pn\le x\\ n<p}} p = \sum_{n\le\sqrt{x}}\sum_{n<p\le\frac{x}{n}} p. \tag{2}$$

By the Abel's summation formula (See Theorem 4.2 of [7]) and the Prime Theorem (See Theorem 3.2 of [8]):

$$\pi(x) = \sum_{i=1}^{k}\frac{a_i\cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n<p\le\frac{x}{n}} p &= \frac{x}{n}\cdot\pi\left(\frac{x}{n}\right) - n\cdot\pi(n) - \int_n^{\frac{x}{n}}\pi(y)dy\\ &= \frac{x^2}{2n^2\ln x} + \sum_{i=2}^{k}\frac{b_i\cdot x^2\cdot\ln^i n}{n^2\cdot\ln^i x} + O\left(\frac{x^2}{n^2\cdot\ln^{k+1} x}\right),\end{aligned} \tag{3}$$

where we have used the estimate $n \le \sqrt{x}$, and all b_i are computable constants.

Note that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, and $\displaystyle\sum_{n=1}^{\infty}\frac{\ln^i n}{n^2}$ is convergent for all $i = 2,\ 3, \cdots, k$. From (2) and (3) we have

$$\begin{aligned}\sum_{n\in A} P(n) &= \sum_{n\le\sqrt{x}}\left(\frac{x^2}{2n^2\ln x} + \sum_{i=2}^{k}\frac{b_i\cdot x^2\cdot\ln^i n}{n^2\cdot\ln^i x} + O\left(\frac{x^2}{n^2\cdot\ln^{k+1} x}\right)\right)\\ &= \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),\end{aligned} \tag{4}$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants.

Now we estimate the summation in set B. Note that for any prime p and positive integer α, $S(p^\alpha) \le \alpha \cdot p$, so from (1) we have

$$\sum_{n\in B} P(n) \quad = \quad \sum_{n\in B} (2S(n)-1) \le \sum_{n\le x} \sqrt{n}\cdot \ln n \ll x^{\frac{3}{2}} \cdot \ln x. \tag{5}$$

Combining (4) and (5) we may immediately deduce the asymptotic formula

$$\sum_{n\le x} P(n) = \sum_{n\in A} P(n) + \sum_{n\in B} P(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2,\ 3, \cdots, k)$ are computable constants. This proves Theorem 1.

Now we prove Theorem 2. For any positive integer $n > 1$, let $\overline{P}(n)$ denotes the largest prime divisor of n. We divide all integers in the interval $[1,\ x]$ into three subsets A, C and D, where A denotes the set of all integers $n \in [1,\ x]$ such that there exists a prime p with $p|n$ and $p > \sqrt{n}$; C denotes the set of all integers $n = n_1 p^2$ in the interval $[1,\ x]$ with $n_1 \le p \le \sqrt{n}$, where p be a prime; And D denotes the set of all integers $n \in [1,\ x]$ with $n \notin A$ and $n \notin C$. It is clear that if $n \in A$, then $P(n) = \overline{P}(n)$ and $\left(P(n) - \overline{P}(n)\right)^2 = 0$. So we have the identity

$$\sum_{n\in A} \left(P(n) - \overline{P}(n)\right)^2 = 0. \tag{6}$$

If $n \in C$, then $P(n) = P(p^2) \ge 2p + 1$. On the other hand, for any real number x large enough, from M.N.Huxley [9] we know that there at least exists a prime in the interval $\left[x,\ x + x^{\frac{7}{12}}\right]$. So we have the estimate

$$2p + 1 \le P(p^2) \le 2p + O\left(p^{\frac{7}{12}}\right). \tag{7}$$

From [3] we also have the asymptotic formula

$$\sum_{n\le x^{\frac{1}{3}}} \sum_{n<p\le\sqrt{\frac{x}{n}}} p^2 = \frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot\frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right). \tag{8}$$

Note that $\overline{P}(n) = p$, if $n = n_1 \cdot p^2 \in C$.

Therefore, from (7) and (8) we have the estimate

$$\begin{aligned}
\sum_{n\in C} \left(P(n) - \overline{P}(n)\right)^2 &= \sum_{n\le x^{\frac{1}{3}}} \sum_{n<p\le\sqrt{\frac{x}{p}}} \left(P(np^2) - \overline{P}(np^2)\right)^2 \\
&= \sum_{n\le x^{\frac{1}{3}}} \sum_{n<p\le\sqrt{\frac{x}{n}}} \left(P(p^2) - p\right)^2 = \sum_{n\le x^{\frac{1}{3}}} \sum_{n<p\le\sqrt{\frac{x}{n}}} \left(p^2 + O\left(p^{\frac{19}{12}}\right)\right) \\
&= \sum_{n\le x^{\frac{1}{3}}} \sum_{n<p\le\sqrt{\frac{x}{n}}} p^2 + O\left(x^{\frac{31}{24}}\right) \\
&= \frac{2}{3}\cdot\zeta\left(\frac{3}{2}\right)\cdot\frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),
\end{aligned} \tag{9}$$

where $\zeta(s)$ is the Riemann zeta-function.

If $n \in D$ and $\left(P(n) - \overline{P}(n)\right)^2 \neq 0$, then $P(p^\alpha) \ll S(p^\alpha) \ll p \cdot \ln p$ and $\overline{P}(p^3) \ll p \cdot \ln p$, so we have the trivial estimate

$$\sum_{n \in D} \left(P(n) - \overline{P}(n)\right)^2 \ll \sum_{3 \leq \alpha \leq \ln x} \sum_{np^\alpha \leq x} p^{\frac{2}{3}} \ll x \cdot \ln x. \tag{10}$$

Combining (6), (9) and (10) we may immediately the asymptotic formula

$$\sum_{n \leq x} \left(P(n) - \overline{P}(n)\right)^2 = \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\overline{P}(n)$ denotes the largest prime divisor of n, and $\zeta(s)$ is the Riemann zeta-function.

This completes the proof of Theorem 2.

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