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Lucas Graceful Labeling for Some Graphs

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Abstract: A *Smarandache-Fibonacci triple* is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Clearly, it is a generalization of *Fibonacci sequence* and *Lucas sequence*. Let G be a (p,q)-graph and $\{S(n)|n \geq 0\}$ a Smarandache-Fibonacci triple. An bijection $f\colon V(G) \rightarrow \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. Particularly, if $S(n), n \geq 0$ is just the Lucas sequence, such a labeling $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_a\}$ $(a \,\epsilon\, N)$ is said to be *Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection on to the set $\{l_1, l_2, \cdots, l_q\}$. Then G is called *Lucas graceful graph* if it admits Lucas graceful labeling. Also an injective function $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_q\}$ is said to be strong Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, ..., l_q\}$. G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling. In this paper, we show that some graphs namely P_n, $P_n^+ - e$, $S_{m,n}$, $F_m@P_n$, $C_m@P_n$, $K_{1,n} \odot 2P_m$, $C_3@2P_n$ and $C_n@K_{1,2}$ admit Lucas graceful labeling and some graphs namely $K_{1,n}$ and F_n admit strong Lucas graceful labeling.

Key Words: Smarandache-Fibonacci triple, super Smarandache-Fibonacci graceful graph, Lucas graceful labeling, strong Lucas graceful labeling.

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§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_n. A cycle of length n is denoted by C_n.G^+ is a graph obtained from the graph G by attaching a pendant vertex to each vertex of G. The concept of graceful labeling was introduced by Rosa [3] in 1967.

A function f is a graceful labeling of a graph G with q edges if f is an injection from

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the vertices of G to the set $\{1, 2, 3, \cdots, q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling was introduced by K.M.Kathiresan and S.Amutha [4]. We call a function, a Fibonacci graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, ...$, and each edge uv is assigned the label $|f(u) - f(v)|$. Based on the above concepts we define the following.

Let G be a (p, q) -graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, \cdots,$. Then G is called Lucas graceful graph if it admits Lucas graceful labeling. Also an injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_q\}$ is said to be strong Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$. Then G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling. In this paper, we show that some graphs namely P_n, $P_n^+ - e$, $S_{m,n}$, $F_m@P_n$, $C_m@P_n$, $K_{1,n} \odot 2P_m$, $C_3@2P_n$ and $C_n@K_{1,2}$ admit Lucas graceful labeling and some graphs namely $K_{1,n}$ and F_n admit strong Lucas graceful labeling. Generally, let $S(n)$, $n \geq 0$ with $S(n) = S(n-1) + S(n-2)$ be a *Smarandache-Fibonacci triple*, where $S(n)$ is the Smarandache function for integers $n \geq 0$. An bijection $f : V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \cdots, S(q)\}$.

§2. Lucas graceful graphs

In this section, we show that some well known graphs are Lucas graceful graphs.

Definition 2.1 *Let G be a (p, q) -graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a, \}$, $(a \in N)$ is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, \cdots,$. Then G is called Lucas graceful graph if it admits Lucas graceful labeling.*

Theorem 2.2 *The path P_n is a Lucas graceful graph.*

Proof Let P_n be a path of length n having $(n + 1)$ vertices namely $v_1, v_2, v_3, \cdots, v_n, v_{n+1}$. Now, $|V(P_n)| = n + 1$ and $|E(P_n)| = n$. Define $f : V(P_n) \to \{l_0, l_1, l_2, \cdots, l_a, \}, a \in N$ by $f(u_i) = l_{i+1}, 1 \leq i \leq n$. Next, we claim that the edge labels are distinct. Let

$$\begin{aligned} E &= \{f_1(v_i v_{i+1}) : 1 \leq i \leq n\} = \{|f(v_i) - f(v_{i+1})| : 1 \leq i \leq n\} \\ &= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|, \cdots, |f(v_n) - f(v_{n+1})|, \} \\ &= \{|l_2 - l_3|, |l_3 - l_4|, \cdots, |l_{n+1} - l_{n+2}|\} = \{l_1, l_2, \cdots, l_n\}. \end{aligned}$$

So, the edges of P_n receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, the path P_n is a Lucas graceful graph. □

Example 2.3 *The graph P_6 admits Lucas graceful Labeling, such as those shown in Fig.1 following.*

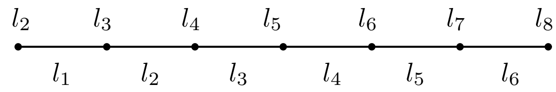

Fig.1

Theorem 2.4 *$P_n^+ - e, (n \geq 3)$ is a Lucas graceful graph.*

Proof Let $G = P_n^+ - e$ with $V(G) = \{u_1, u_2, \cdots, u_{n+1}\} \bigcup \{v_2, v_3, \cdots, v_{n+1}\}$ be the vertex set of G. So, $|V(G)| = 2n + 1$ and $|E(G)| = 2n$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a, \}, a \epsilon N$, by

$$f(u_i) = l_{2i-1}, 1 \leq i \leq n+1 \quad \text{and} \quad f(v_j) = l_{2(j-1)}, 2 \leq j \leq n+1.$$

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \{f_1(u_i u_{i+1}) : 1 \leq i \leq n\} = \{|f(u_i) - f(u_{i+1})| : 1 \leq i \leq n\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \cdots, |f(u_n) - f(u_{n+1})|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|, \cdots, |l_{2n-1} - l_{2n+1}|\} = \{l_2, l_4, \cdots, l_{2n}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f_1(u_i v_j) : 2 \leq i,\ j \leq n\} \\ &= \{|f(u_2) - f(v_2)|, |f(u_3) - f(v_3)|, \cdots, |f(u_{n+1}) - f(v_{n+1})|\} \\ &= \{|l_3 - l_2|, |l_5 - l_4|, \cdots, |l_{2n+1} - l_{2n}|\} = \{l_1, l_3, \cdots, l_{2n-1}\}. \end{aligned}$$

Now, $E = E_1 \cup E_2 = \{l_1, l_3, \cdots, l_{2n-1}, l_{2n}\}$. So, the edges of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $P_n^+ - e, (n \geq 3)$ is a Lucas graceful graph. □

Example 2.5 The graph $P_8^+ - e$ admits Lucas graceful labeling, such as thsoe shown in Fig.2.

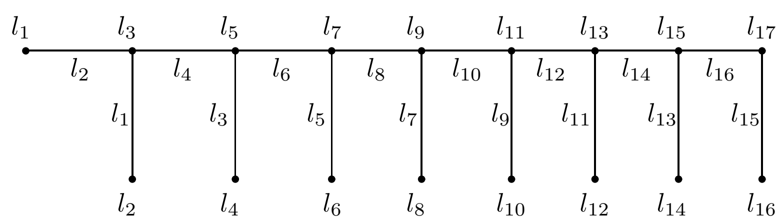

Fig.2

Definition 2.6([2]) *Denote by $S_{m,n}$ such a star with n spokes in which each spoke is a path of length m.*

Theorem 2.7 *The graph $S_{m,n}$ is a Lucas graceful graph when m is odd and $n \equiv 1, 2 (mod\ 3)$.*

Proof Let $G = S_{m,n}$ and let $V(G) = \left\{u_j^i : 1 \leq i \leq m \text{ } and \text{ } 1 \leq j \leq n\right\}$ be the vertex set of $S_{m,n}$. Then $|V(G)| = mn + 1$ and $|E(G)| = mn$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a, \}, a \in N$ by

$f(u_0) = l_0$ for $i = 1, 2, \cdots, m-2$ and $i \equiv 1(mod\ 2)$;
$f\left(u_j^i\right) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$ for $i = 1, 2, \cdots, m-1$ and $i \equiv 0(mod\ 2)$;
$f\left(u_j^i\right) = l_{ni+2-2j}, 1 \leq j \leq n$ and for $s = 1, 2, \cdots, \dfrac{n}{3}$,
$f\left(u_j^m\right) = l_{n(m-1)+2(j+1)-3s}, 3s-2 \leq j \leq 3s.$

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{f_1\left(u_0\ u_1^i\right)\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|f\left(u_0\right) - f\left(u_1^i\right)\right|\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{n(i-1)+1}\right|\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+1}\right\} \\ &= \left\{l_1, l_{2n+1}, l_{4n+1}, \cdots, l_{n(m-1)+1}\right\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{f_1\left(u_0\ u_1^i\right)\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{\left|f\left(u_0\right) - f\left(u_1^i\right)\right|\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{\left|l_0 - l_{ni}\right|\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{l_{ni}\right\} = \left\{l_{2n}, l_{4n}, \cdots, l_{n(m-1)}\right\} \end{aligned}$$

$$\begin{aligned} E_3 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{f_1\left(u_j^i u_{j+1}^i\right) : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|f\left(u_j^i\right) - f\left(u_{j+1}^i\right)\right| : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}\right| : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{n(i-1)+2j} : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{n(i-1)+2}, l_{n(i-1)+4}, \cdots, : l_{n(i-1)+2(n-1)}\right\} \\ &= \left\{l_2, l_{2n+2}, ..., l_{n(m-3)+2}\right\} \cup \left\{l_4, l_{2n+4}, \cdots, l_{n(m-3)+4}\right\} \cup \cdots \\ &\quad \cup \left\{l_{2n-2}, l_{4n-2}, ..., l_{n(m-3)+2n-2}\right\}, \end{aligned}$$

$$\begin{aligned}
E_4 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{f_1\left(u_j^i\ u_{j+1}^i\right) : 1\le j\le n-1\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|f\left(u_j^i\right)-f\left(u_{j+1}^i\right)\right| : 1\le j\le n-1\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|l_{ni-2j+2}-l_{ni-2j}\right| : 1\le j\le n-1\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{ni-2j+1} : 1\le j\le n-1\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{ni-1}, l_{ni-3}, \cdots, l_{ni-(2n-3)}\right\}\\
&= \left\{l_{2n-1}, l_{2n-3}, \cdots, l_3, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, l_{n(m-1)-1}, \cdots, l_{n(m-1)-(2n-3)}\right\}.
\end{aligned}$$

For $n\equiv 1(mod\ 3)$, let

$$\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{f_1\left(u_j^m\ u_{j+1}^m\right) : 3s-2\le j\le 3s-1\right\}\\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|f\left(u_j^m\right)-f\left(u_{j+1}^m\right)\right| : 3s-2\le j\le 3s-1\right\}\\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|l_{n(m-1)+2j-3s+2}-l_{n(m-1)+2j-3s+4}\right| : 3s-2\le j\le 3s-1\right\}\\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{l_{n(m-1)+2j-3s+2} : 3s-2\le j\le 3s-1\right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{l_{n(m-1)+3s-1}, l_{n(m-1)+3s+1}\right\}\\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{n\ m-2}, l_{mn}\right\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s=1,2,\cdots,\dfrac{n-1}{3}$. Let

$$\begin{aligned}
E_6 &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|f_1\left(u_{3s}^m\ u_{3s+1}^m\right)\right|\right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|f\left(u_{3s}^m\right)-f\left(u_{3s+1}^m\right)\right|\right\}\\
&= \left\{\left|f\left(u_3^m\right)-f\left(u_4^m\right)\right|, \left|f\left(u_6^m\right)-f\left(u_7^m\right)\right|, \left|f\left(u_9^m\right)-f\left(u_{10}^m\right)\right|, \cdots, \left|f\left(u_{n-1}^m\right)-f\left(u_n^m\right)\right|\right\}\\
&= \left\{\left|l_{n(m-1)+5}-l_{n(m-1)+4}\right|, \left|l_{n(m-1)+8}-l_{n(m-1)+7}\right|, \cdots, \left|l_{n(m-1)+n+1}-l_{n(m-1)+n}\right|\right\}\\
&= \left\{l_{n(m-1)+3}, l_{n(m-1)+6}, \cdots, l_{n(m-1)+n-1}\right\} = \left\{l_{n(m-1)+3}, l_{n(m-1)+6}, \cdots, l_{nm-1}\right\}.
\end{aligned}$$

For $n \equiv 2(mod\ 3)$, let

$$
\begin{aligned}
E_5' &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ f_1\left(u_j^m\ u_{j+1}^m\right) : 3s-2 \le j \le 3s-1 \right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ \left| f\left(u_j^m\right) - f\left(u_{j+1}^m\right) \right| : 3s-2 \le j \le 3s-1 \right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ \left| l_{n(m-1)+2j-3s+2} - l_{n(m-1)+2j-3s+4} \right| : 3s-2 \le j \le 3s-1 \right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ l_{n(m-1)+2j-3s+3} : 3s-2 \le j \le 3s-1 \right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ l_{n(m-1)+3s-1}, l_{n(m-1)+3s+1} \right\} \\
&= \left\{ l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{n(m-1)+n-2}, l_{n(m-1)+n} \right\}.
\end{aligned}
$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, 3, ..., \dfrac{n-1}{3}$.

$$
\begin{aligned}
Let\ E_6' &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ f_1\left(u_{3s}^m\ u_{3s+1}^m\right) \right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ \left| f\left(u_{3s}^m\right) - f\left({}^{m}_{3s+1}\right) \right| \right\} \\
&= \left\{ \left| f\left(u_3^m\right) - f\left(u_4^m\right) \right|, \left| f\left(u_6^m\right) - f\left(u_7^m\right) \right|, \left| f\left(u_9^m\right) - f\left(u_{10}^m\right) \right|, \cdots, \left| f\left(u_{n-1}^m\right) - f\left(u_n^m\right) \right| \right\} \\
&= \left\{ \left| l_{n(m-1)+5} - l_{n(m-1)+4} \right|, \left| l_{n(m-1)+8} - l_{n(m-1)+7} \right|, \cdots, \left| l_{n(m-1)+n+1} - l_{n(m-1)+n} \right| \right\} \\
&= \left\{ l_{n(m-1)+3}, l_{n(m-1)+6}, \cdots, l_{nm-1} \right\}.
\end{aligned}
$$

Now, $E = \bigcup\limits_{i=1}^{6} E_i$ if $n \equiv 1(mod\ 3)$ and $E = \left(\bigcup\limits_{i=1}^{6} E_i\right) \bigcup E_5' \bigcup E_6'$ if $n \equiv 2(mod\ 3)$. So the edges of $S_{m,n}$ (when m is odd and $n \equiv 1, 2(mod\ 3)$), receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $S_{m,n}$ is a Lucas graceful graph if m is odd, $n \equiv 1, 2(mod\ 3)$. □

Example 2.8 The graphs $S_{5,4}$ and $S_{5,5}$ admit Lucas graceful labeling, such as those shown in Fig.3 and Fig 4.

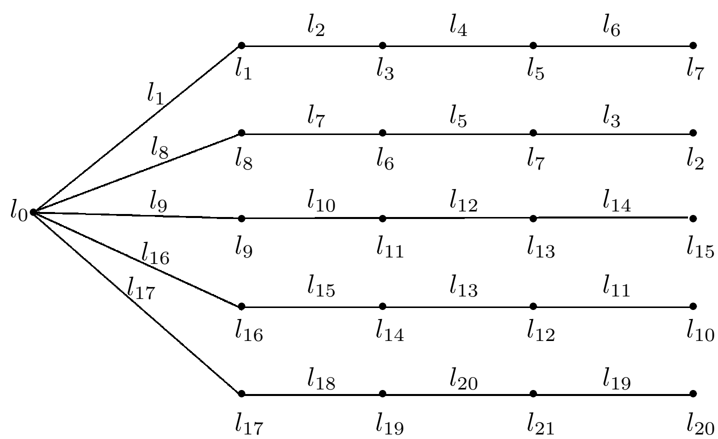

Fig.3

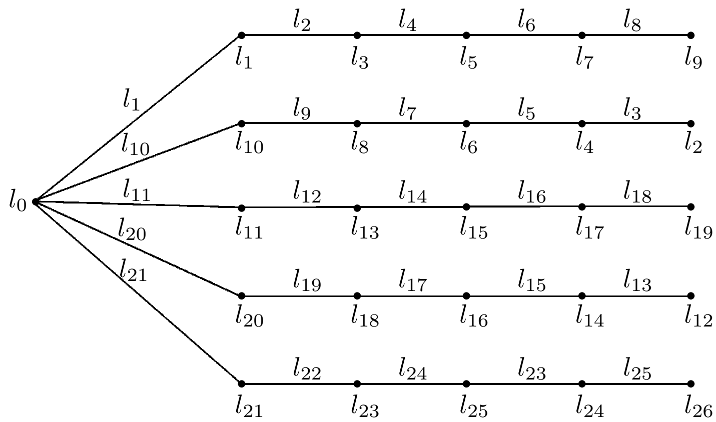

Fig.4

Definition 2.9([2]) *The graph $G = F_m@P_n$ consists of a fan F_m and a path P_n of length n which is attached with the maximum degree of the vertex of F_m.*

Theorem 2.10 *$F_m@P_n$ is a Lucas graceful labeling when $n \equiv 1, 2$ $(mod\ 3)$.*

Proof Let $v_1, v_2, ..., v_m, v_{m+1}$ and u_0 be the vertices of a fan F_m and $u_1, u_2, \cdots, u_n$ be the vertices of a path P_n. Let $G = F_m@P_n$. Then $|V(G)| = m + n + 2$ and $|E(G)| = 2m + n + 1$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a,\}, a\ \epsilon\ N$, by $f(u_0) = l_0$, $f(v_i) = l_{2i-1}, 1 \leq i \leq m + 1$. For $s = 1, 2, \cdots, \dfrac{n-1}{3}$ *or* $\dfrac{n-2}{3}$ according as $n \equiv 1(mod\ 3)$ or $n \equiv 2(mod\ 3)$, $f(u_j) = l_{2m+2j-3s+3}, 3s - 2 \leq j \leq 3s$.

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \{f_1(v_i\ v_{i+1}) : 1 \leq i \leq m\} = \{|f(v_i) - f\,(v_{i+1})| : 1 \leq i \leq m\} \\ &= \{|l_{2i-1} - l_{2i+1}| : 1 \leq i \leq m\} \\ &= \{l_{2i} : 1 \leq i \leq m\} = \{l_2, l_4, \cdots, l_{2m}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f_1\,(u_0 v_i) : 1 \leq i \leq m+1\} = \{|f(u_0) - f(v_i)|\, 1 \leq i \leq m+1\} \\ &= \{|l_0 - l_{2i-1}| : 1 \leq i \leq m+1\} \\ &= \{l_{2i-1} : 1 \leq i \leq m+1\} = \{l_1, l_3, \cdots, l_{2m+1}\} \end{aligned}$$

and

$$E_3 = \{f_1\,(u_0 u_1)\} = \{|f(u_0) - f(u_1)|\} = \{|l_0 - l_{2m+2}|\} = \{l_{2m+2}\}$$

For $s = 1, 2, 3, \cdots, \dfrac{n-1}{3}$ and $n \equiv 1(mod\ 3)$, let

$$\begin{aligned}
E_4 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1\left(u_j, u_{j+1}\right) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_j) - f(u_{j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} (l_{2m+2j+4-3s} : 3s-2 \leq j \leq 3s-1) \\
&= \{l_{2m+2j-2} : 4 \leq j \leq 5\} \bigcup \{l_{2m+2j-5} : 7 \leq j \leq 8\} \bigcup \cdots \\
&\quad \bigcup \{l_{2m+2j-n+4} : n-3 \leq j \leq n-2\} \\
&= \{l_{2m+6}, l_{2m+8}\} \cup \{l_{2m+9}, l_{2m+11}\} \bigcup \cdots \bigcup \{l_{2m+n-2}, l_{2m+n}\} \\
&= \{l_{2m+6}, l_{2m+8}, l_{2m+9}, l_{2m+11}, \cdots, l_{2m+n-2}, l_{2m+n}\}
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, 3, \cdots, \dfrac{n-1}{3}$, $n \equiv 1(mod\ 3)$. Let

$$\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1\left(u_j\ u_{j+1}\right) : j = 3s\} = \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_j) - f(u_{j+1})| : j = 3s\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : j = 3s\} \\
&= \{|l_{2m+2j} - l_{2m+2j-1}| : j = 3\} \cup \{|l_{2m+2j-3} - l_{2m+2j-4}| : j = 6\} \bigcup \cdots \\
&\quad \bigcup \{|l_{2m+2j} - l_{2m+2j-1}| : j = n-1\} \\
&= \{l_{2m+2j-2} : j = 3\} \cup \{l_{2m+2j-5} : j = 6\} \cup, \cdots, \cup \{l_{2m+2j-n+3} : j = n-1\} \\
&= \{l_{2m+4}, l_{2m+7}, \cdots, l_{2m+n+1}\}.
\end{aligned}$$

For $s = 1, 2, 3, \cdots, \dfrac{n-2}{3}$ and $n \equiv 2(mod\ 3)$, let

$$\begin{aligned}
E_4^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_j\ u_{j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_j) - f(u_{j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : 3s-2 \leq j \leq 3s-1\}
\end{aligned}$$

$$\begin{aligned}
&= \bigcup_{s=1}^{\frac{n-2}{3}} (l_{2m+2j+4-3s} : 3s-2 \leq j \leq 3s-1) \\
&= \{l_{2m+2j-2} : 4 \leq j \leq 5\} \bigcup \{l_{2m+2j-5} : 7 \leq j \leq 8\} \bigcup \cdots \\
&\quad \bigcup \{l_{2m+2j-n+4} : n-3 \leq j \leq n-2\} \\
&= \{l_{2m+6}, l_{2m+8}\} \bigcup \{l_{2m+9}, l_{2m+11}\} \bigcup \cdots \bigcup \{l_{2m+n-2}, l_{2m+n}\} \\
&= \{l_{2m+6}, l_{2m+8,2m+9}, l_{2m+11}, \cdots, l_{2m+n-2}, l_{2m+n}\}
\end{aligned}$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, 3, ..., \frac{n-2}{3}$, $n \equiv 2(mod\ 3)$. Let

$$\begin{aligned}
E_5^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_j, u_{j+1}) : j = 3s\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_j) - f(u_{j+1})| : j = 3s\} = \bigcup_{s=1}^{\frac{n-2}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : j = 3s\} \\
&= \{|l_{2m+2j} - l_{2m+2j-1}| : j = 3\} \bigcup \{|l_{2m+2j-3} - l_{2m+2j-4}| : j = 6\} \bigcup \cdots \\
&\quad \bigcup \{|l_{2m+2j-n+4} - l_{2m+2j-n+5}| : j = n-1\} \\
&= \{l_{2m+2j-2} : j = 3\} \cup \{l_{2m+2j-5} : j = 6\} \bigcup \cdots \bigcup \{l_{2m+2j-(n-3)} : j = n-1\} \\
&= \{l_{2m+4}, l_{2m+7}, ..., l_{2m+n+1}\}.
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{5} E_i$ if $n \equiv 1(mod\ 3)$ and $E = \left(\bigcup_{i=1}^{5} E_i\right) \bigcup E_4^{'} \bigcup E_5^{'}$ if $n \equiv 2(mod\ 3)$. So, the edges of $F_m@P_n$ (when$n \equiv 1, 2(mod\ 3)$) are the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $G = F_m@P_n$ (if $n \equiv 1, 2(mod\ 3)$) is a Lucas graceful labeling. □

Example 2.11 The graph $F_5@P_4$ admits a Lucas graceful labeling shown in Fig.5.

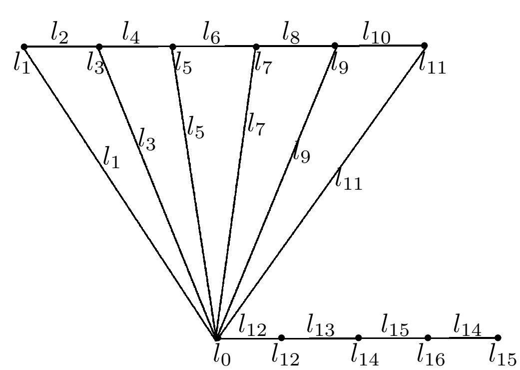

Fig.5

Definition 2.12 ([2]) *The Graph $G = C_m@P_n$ consists of a cycle C_m and a path of P_n of length n which is attached with any one vertex of C_m.*

Theorem 2.13 *The graph $C_m@P_n$ is a Lucas graceful graph when $m \equiv 0(mod\ 3)$ and $n = 1, 2(mod\ 3)$.*

Proof Let $G = C_m@P_n$ and let $u_1, u_2, \cdots, u_m$ be the vertices of a cycle C_m and $v_1, v_2, \cdots, v_n, v_{n+1}$ be the vertices of a path P_n which is attached with the vertex $(u_1 = v_1)$ of C_m. Let $V(G) = \{u_1 = v_1\} \cup \{u_2, u_3, \cdots, u_m\} \cup \{v_2, v_3, ..., v_n, v_{n+1}\}$ be the vertex set of G. So, $|V(G)| = m + n$ and $|E(G)| = m + n$. Define $f : V(G) \to \{l_0, l_1, \cdots, l_a\},\ a \epsilon N$ by $f(u_1) = f(v_1) = l_0$; $f(u_i) = l_{2i-3s}, 3s - 1 \leq j \leq 3s + 1$ for $s = 1, 2, 3, \cdots, \dfrac{m}{3}$, $i = 2, 3, \cdots, m$; $f(v_j) = l_{m+2j-3r}, 3r - 1 \leq j \leq 3r + 1$ for $r = 1, 2, \cdots, \dfrac{n+1}{3}$ and $j = 2, 3, \cdots, n + 1$.

We claim that the edge labels are distinct. Let

$$E_1 = \{f_1(u_1\ u_2)\} = \{|f(u_1) - f(u_2)|\} = (|l_0 - l_1|) = \{l_1\},$$

$$\begin{aligned} E_2 &= \bigcup_{s=1}^{\frac{m}{3}} \{f_1(u_i\ u_{i+1}) : 3s - 1 \leq i \leq 3s \ and\ u_{m+1} = u_1\} \\ &= \bigcup_{s=1}^{\frac{m}{3}} \{f_1(u_i) - f(u_{i+1}) : 3s - 1 \leq i \leq 3s \ and\ u_{m+1} = u_1\} \\ &= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|, ..., |f(u_m) - f(u_{m+1})|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|, |l_4 - l_6|, |l_6 - l_8|, \cdots, |l_m - l_0|\} \\ &= \{l_2, l_4, l_5, l_7, \cdots, l_m\} \end{aligned}$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, ..., \dfrac{m}{3} - 1$. Let

$$\begin{aligned} E_3 &= \bigcup_{s=1}^{\frac{m}{3}-1} \{f_1(u_{3s+1}\ u_{3s+2})\} = \bigcup_{s=1}^{\frac{m}{3}-1} \{|f(u_{3s+1}) - f(u_{3s+2})|\} \\ &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \cdots, |f(u_{m-2}) - f(u_{m-1})|\} \\ &= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{m-1} - l_{m-2}|\} \\ &= \{l_3, l_6, \cdots, l_{m-3}\}, \end{aligned}$$

$$\begin{aligned} E_4 &= \{f_1(v_1\ v_2)\} = \{|f(v_1) - f(v_2)|\} = \{|l_0 - l_{m+4-3}|\} \{|l_0 - l_{m+4-3}|\} \\ &= \{|l_0 - l_{m+1}|\} = \{|l_0 - l_{m+1}|\} = \{l_{m+1}\}. \end{aligned}$$

For $n \equiv 1(mod\ 3)$, let

$$\begin{aligned} E_5 &= \bigcup_{r=1}^{\frac{n-1}{3}} \{f_1(v_j\ v_{j+1}) : 3r - 1 \leq j \leq 3r\} \\ &= \bigcup_{r=1}^{\frac{n-1}{3}} \{|f(v_j) - f(v_{j+1})| : 3r - 1 \leq j \leq 3r\} \end{aligned}$$

$$
\begin{aligned}
&= \{|f(v_2)-f(v_3)|, |f(v_3)-f(v_4)|, \cdots, |f(v_{n-1})-f(v_n)|\} \\
&= \{|l_{m+4-3}-l_{m+6-3}|, |l_{m+6-3}-l_{m+8-3}|, |l_{m+10-6}-l_{m+12-6}|, |l_{m+12-6}-l_{m+14-6}|, \\
&\quad \cdots, |l_{m+2n-2-n+1}-l_{m+2n-n+1}|\} \\
&= \{|l_{m+1}-l_{m+3}|, |l_{m+3}-l_{m+5}|, |l_{m+4}-l_{m+6}|, |l_{m+6}-l_{m+8}|, \cdots, |l_{m+n-1}-l_{m+n+1}|\} \\
&= \{l_{m+2}, l_{m+4}, l_{m+5}, l_{m+7}, \cdots, l_{m+n}\}.
\end{aligned}
$$

We calculate the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop and $r = 1, 2, \cdots, \dfrac{n-1}{3}$. Let

$$
\begin{aligned}
E_6 &= \bigcup_{r=1}^{\frac{n-1}{3}} \{f_1(v_{3r+1}\ v_{3r+2})\} = \bigcup_{r=1}^{\frac{n-1}{3}} \{|f(v_{3r+1})-f(v_{3r+2})|\} \\
&= \{|f(v_4)-f(v_5)|, |f(v_7)-f(v_8)|, \cdots, |f(v_{n-2})-f(v_{n-1})|\} \\
&= \{|l_{m+8-3}-l_{m+10-6}|, |l_{m+14-6}-l_{m+16-9}|, \cdots, |l_{m+2n-4-n+2}-l_{m+2n-2-n+1}|\} \\
&= \{|l_{m+5}-l_{m+4}|, |l_{m+8}-l_{m+7}|, \cdots, |l_{m+n-2}-l_{m+n}|\} \\
&= \{l_{m+3}, l_{m+6}, l_{m+9}, \cdots, l_{m+n-1}\}
\end{aligned}
$$

For $n \equiv 2(mod\ 3)$, let

$$
\begin{aligned}
E_5' &= \bigcup_{r=1}^{\frac{n-1}{3}} \{f_1(v_j\ v_{j+1}) : 3r-1 \le j \le 3r\} = \bigcup_{r=1}^{\frac{n-1}{3}} \{|f(v_j)-f(v_{j+1})| : 3r-1 \le j \le 3r\} \\
&= \{|f(v_2)-f(v_3)|, |f(v_3)-f(v_4)|, \cdots, |f(v_{n-1})-f(v_n)|\} \\
&= \{|l_{m+4-3}-l_{m+6-3}|, |l_{m+6-3}-l_{m+8-3}|, |l_{m+10-6}-l_{m+12-6}|, |l_{m+12-6}-l_{m+14-6}|, \\
&\quad \cdots, |l_{m+2n-2-2n+1}-l_{m+2n-n+1}|\} \\
&= \{l_{m+2}, l_{m+4}, l_{m+5}, l_{m+7}, ..., l_{m+n}\}.
\end{aligned}
$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop and $r = 1, 2, \cdots, \dfrac{n-2}{3}$. Let

$$
\begin{aligned}
E_6' &= \bigcup_{r=1}^{\frac{n-2}{3}} \{f_1(v_{3r+1}\ v_{3r+2})\} = \bigcup_{r=1}^{\frac{n-2}{3}} \{|f(v_{3r+1})-f(v_{3r+2})|\} \\
&= \{|f(v_4)-f(v_5)|, |f(v_7)-f(v_8)|, \cdots, |f(v_{n-2})-f(v_{n-1})|\} \\
&= \{|l_{m+8-3}-l_{m+10-6}|, |l_{m+14-6}-l_{m+16-9}|, \cdots, |l_{m+2n-4-n+2}-l_{m+2n-2-n+1}|\} \\
&= \{|l_{m+5}-l_{m+4}|, |l_{m+8}-l_{m+7}|, \cdots, |l_{m+n-2}-l_{m+n}|\} \\
&= \{l_{m+3}, l_{m+6}, l_{m+9}, \cdots, l_{m+n-1}\}
\end{aligned}
$$

Now, $E = \bigcup\limits_{i=1}^{6} E_i$ if $n \equiv 1(mod\ 3)$ and $E = \left(\bigcup\limits_{i=1}^{4} E_i\right) \bigcup E_5' \bigcup E_6'$ if $n \equiv 2(mod\ 3)$. So, the edges of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $G = C_m@P_n$ is a Lucas graceful graph when $m \equiv 0(mod\ 3)$ and $n \equiv 1, 2(mod\ 3)$. □

Example 2.14 The graph $C_9@P_7$ admits a Lucas graceful labeling, such as those shown in Fig.6.

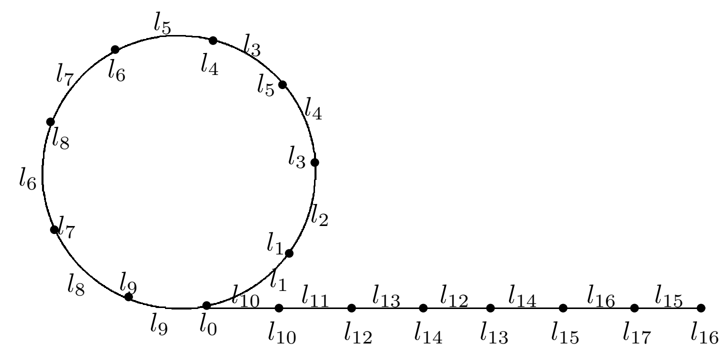

Fig.6

Definition 2.15 *The graph $K_{1,n} \odot 2P_m$ means that 2 copies of the path of length m is attached with each pendent vertex of $K_{1,n}$.*

Theorem 2.16 *The graph $K_{1,n} \odot 2P_m$ is a Lucas graceful graph.*

Proof Let $G = K_{1,n} \odot 2P_m$ with $V(G) = \{u_i : 0 \le i \le n\} \bigcup \Big\{v_{ij}^{(1)}, v_{i,j}^{(2)} : 1 \le i \le n, 1 \le j \le m-1\}$ and $E(G) = \{u_0\ u_i : 1 \le i \le n\} \cup \Big\{u_i\ v^{(1)_{i,j}}, u_i\ v_{i,j}^{(2)} : 1 \le i \le n \text{ and } 1 \le j \le m-1\Big\} \cup \Big\{v_{i,j}^{(1)}\ v_{i,j+1}^{(1)}, v_{i,j}^{(2)}\ v_{i,j+1}^{(2)} : 1 \le i \le n \text{ and } 1 \le j \le m-1\Big\}$. Thus $|V(G)| = 2mn + n + 1$ and $|E(G)| = 2mn + n$.

For $i = 1, 2, \cdots, n$, define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, a \in N$, by $f(u_0) = l_0$ $f(u_i) = l_{(2m+1)(i-1)+2}$; $f(v_{i,j}^{(1)}) = l_{(2m+1)(i-1)+2j+1}$, $1 \le j \le m$ and $f(v_{i,j}^{(2)}) = l_{(2m+1)(i-1)+2j+2}$, $1 \le j \le m$.

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{i=1}^{n} \{f_1(u_0\ u_i)\} = \bigcup_{i=1}^{n} \{|f(u_0) - f(u_i)|\} \\ &= \bigcup_{i=1}^{n} \{|l_0 - l_{(2m+1)(i-1)+2}|\} = \bigcup_{i=1}^{n} \{l_{(2m+1)(i-1)+2}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{i=1}^{n} \Big\{f_1(u_i v_{i,1}^{(1)}), f_1(u_i v_{i,1}^{(2)})\Big\} \\ &= \bigcup_{i=1}^{n} \Big\{\Big|f(u_i) - f(v_{i,1}^{(1)}\Big|, \Big|f(u_i) - f(v_{i,1}^{(2)})\Big|\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{i=1}^{n} \left\{ \left| l_{(2m+1)(i-1)+2} - l_{(2m+1)(i-1)+3} \right|, \left| l_{(2m+1)(i-1)+2} - l_{(2m+1)(i-1)+4} \right| \right\} \\
&= \bigcup_{i=1}^{n} \left\{ l_{(2m+1)(i-1)+1}, l_{(2m+1)(i-1)+3} \right\} \\
&= \{l_1, l_3\} \cup \{l_{2m+2}, l_{2m+4}\} \cup \{l_{2mn+n-2m+1}, l_{2mn+n-2m+3}\} \\
&= \{l_1, l_{2m+2}, \cdots, l_{2mn+n-2m+1}, l_3, l_{2m+4}, \cdots, l_{2mn+n-2m+3}\},
\end{aligned}
$$

$$
\begin{aligned}
E_3 &= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ f_1(v_{i,j}^{(1)} \ v_{i,j+1}^{(1)} \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ \left| f(v_{i,j}^{(1)}) - f(v_{i,j+1}^{(1)} \right| \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ \left| l_{(2m+1)(i-1)+2j+1} - l_{(2m+1)(i-1)+2j+3} \right| \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ l_{(2m+1)(i-1)+2j+2} \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ l_{(2m+1)(i-1)+4}, l_{(2m+1)(i-1)+6}, \cdots, l_{(2m+1)(i-1)+2m} \right\} \\
&= \{l_4, l_6, \cdots, l_{2m}\} \cup \left\{ l_{(2m+1)+4}, l_{(2m+1)+6}, \cdots, l_{(2m+1)(i-1)+2m} \right\} \bigcup \\
&\quad \cdots \bigcup \left\{ l_{(2m+1)(n-1)+4}, l_{(2m+1)(n-1)+6}, \cdots, l_{(2m+1)(n-1)+2m} \right\} \\
&= \left\{ l_4, \cdots, l_{2m}, l_{2m+5}, \cdots, l_{4m+1}, \cdots, l_{(2m+1)(n-1)+4}, l_{(2m+1)(n-1)+6}, \cdots, l_{2mn+n-1} \right\},
\end{aligned}
$$

$$
\begin{aligned}
E_4 &= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ f_1(v_{i,j}^{(2)} \ v_{i,j+1}^{(2)} \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ \left| f(v_{i,j}^{(2)}) - f(v_{i,j+1}^{(2)} \right| \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ \left| l_{(2m+1)(i-1)+2j+2} - l_{(2m+1)(i-1)+2j+4} \right| \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{m-1} \left\{ l_{(2m+1)(i-1)+2j+3} \right\} \right\} \\
&= \bigcup_{i=1}^{n} \left\{ l_{(2m+1)(i-1)+5}, l_{(2m+1)(i-1)+7}, \cdots, l_{(2m+1)(i-1)+2m+1} \right\} \\
&= \{l_5, \cdots, l_{2m+1}\} \bigcup \{l_{2m+1+5}, l_{2m+1+7}, \cdots, l_{2m+1+2m+1}\} \\
&\quad \bigcup \left\{ l_{(2m+1)(n-1)+5}, l_{(2m+1)(n-1)+7}, \cdots, l_{(2m+1)(n-1)+(2m+1)} \right\} \\
&= \left\{ l_5, \cdots, l_{2m+1}, l_{2m+6}, \cdots, l_{4m+1}, \cdots, l_{(2m+1)(n-1)+5}, l_{(2m+1(n-1)+7}, \cdots, l_{(2m+1)+n} \right\}.
\end{aligned}
$$

Now, $E = \bigcup_{i=1}^{4} E_i = \{l_1, l_2, ..., l_{(2m+1)n}\}$. So, the edge labels of G are distinct. Therefore, f is a Lucas graceful labeling. Hence, $G = K_{1,n} \odot 2P_m$ is a Lucas graceful labeling. $\Box$

Example 2.17 The graph $K_{1,4} \odot 2P_4$ admits Lucas graceful labeling, such as those shown in Fig.7.

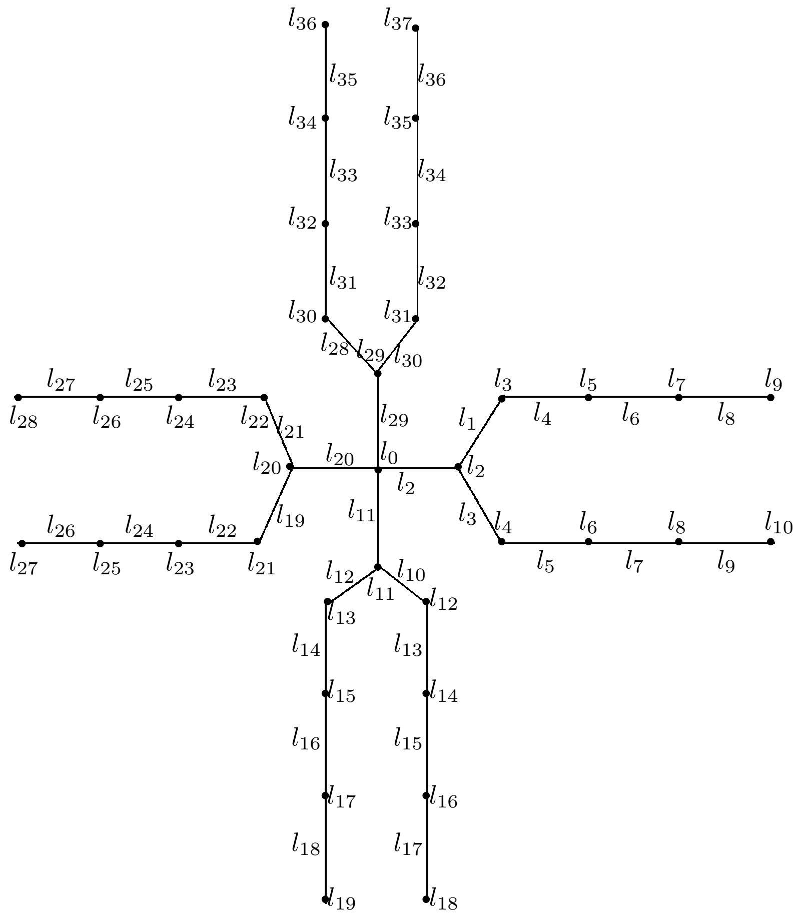

Fig.7

Theorem 2.18 *The graph $C_3@2P_n$ is Lucas graceful graph when $n \equiv 1(mod\ 3)$.*

Proof Let $G = C_3@2P_n$ with $V(G) = \{w_i : 1 \leq i \leq 3\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\}$ and the vertices w_2 and w_3 of C_3 are identified with v_1 and u_1 of two paths of length n respectively. Let $E(G) = \{w_i w_{i+1} : 1 \leq i \leq 2\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n\}$ be the edge set of G. So, $|V(G)| = 2n + 3$ and $|E(G)| = 2n + 3$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, a \in N$ by $f(w_1) = l_{n+4}$; $f(u_i) = l_{n+3-i}, 1 \leq i \leq n+1$; $f(v_j) = l_{n+4+2j-3s}$, $3s - 2 \leq j \leq 3s$ for $s = 1, 2, ..., \dfrac{n-1}{3}$ and $f(v_j) = l_{n+4+2j-3s}$ $3s - 2 \leq j \leq 3s - 1$ for $s = \dfrac{n-1}{3} + 1$.

We claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \bigcup_{i=1}^{n} \{f_1(u_i u_{i+1}\} = \bigcup_{i=1}^{n} \{|f(u_i) - f(u_{i+1})|\} \\
&= \bigcup_{i=1}^{n} \{|l_{n+3-i} - l_{n+3-i-1}|\} = \bigcup_{i=1}^{n} \{|l_{n+3-i} - l_{n+2-i}|\} \\
&= \bigcup_{i=1}^{n} \{l_{n+1-i}\} = \{l_n, l_{n-1}, \cdots, l_1\}, \\
E_2 &= \{f_1(u_1 w_1), f_1(w_1 v_1), f_1(v_1 u_1)\} \\
&= \{|f(u_1) - f(w_1)|, |f(w_1 - f(v_1)|, |f(v_1) - f(u_1)|\} \\
&= \{|l_{n+2} - l_{n+4}|, |l_{n+4} - l_{n+3}|, |l_{n+3} - l_{n+2}|\} = \{l_{n+3}, l_{n+2}, l_{n+1}\}.
\end{aligned}$$

For $s = 1, 2, \cdots, \dfrac{n-1}{3}$, let

$$\begin{aligned}
E_3 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(v_j v_{j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(v_j) - f(v_{j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \cup \{|f(v_4) - f(v_5)|, |f(v_5) - f(v_6)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(v_{n-3}) - f(v_{n-2})|, |f(v_{n-2}) - f(v_{n-1})|\} \\
&= \{|l_{n+3} - l_{n+5}|, |l_{n+5} - l_{n+7}|\} \cup \{|l_{n+6} - l_{n+8}|, |l_{n+8} - l_{n+10}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{2n-1} - l_{2n+1}|, |l_{2n+1} - l_{2n+3}|\} \\
&= \{l_{n+4}, l_{n+6}\} \bigcup \{l_{n+7}, l_{n+9}\} \bigcup \cdots \bigcup \{l_{2n}, l_{2n+2}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $1 \leq s \leq \dfrac{n-1}{3}$. Let

$$\begin{aligned}
E_4 &= \{f_1(v_j v_{j+1}) : j = 3s\} = \{|f(v_j) - f(v_{j+1})| : j = 3s\} \\
&= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, \cdots, |f(v_{n-1}) - f(v_n)|\} \\
&= \{|l_{n+7} - l_{n+6}|, |l_{n+10} - l_{n+9}|, \cdots, |l_{2n+3} - l_{2n+2}|\} = \{l_5, l_8, \cdots, l_{2n+1}\}.
\end{aligned}$$

For $s = \dfrac{n-1}{3} + 1$, let

$$\begin{aligned}
E_5 &= \{f_1(v_j v(_{j+1}) : j = 3s-2\} = \{|f(v_j) - f(v_{j+1})| : j = n\} \\
&= \{|f(v_n) - f(v_{n+1})|\} = \{|l_{n+4+2n-n-2} - l_{n+4+2n+2-n-2}|\} \\
&= \{|l_{2n+2} - l_{2n+4}|\} = \{l_{2n+3}\}.
\end{aligned}$$

Now, $E = \bigcup\limits_{s=1}^{5} E_i = \{l_1, l_2, ..., l_{2n+3}\}$. So, the edge labels of G are distinct. Therefore, f is a Lucas graceful labeling. Hence, $G = C_3 @ 2P_n$ is a Lucas graceful graph if $n \equiv 1(mod\ 3)$. $\square$

Example 2.19 The graph $C_3@2P_4$ admits Lucas graceful labeling shown in Fig.8.

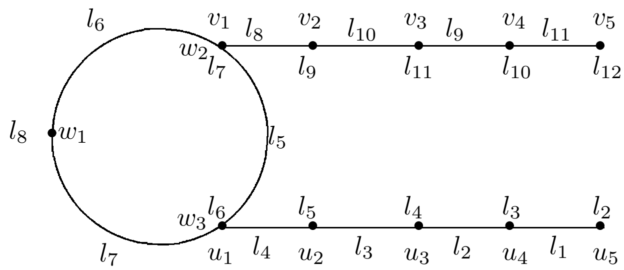

Fig.8

Theorem 2.20 *The graph $C_n@K_{1,2}$ is a Lucas graceful graph if $n \equiv 1(mod\ 3)$.*

Proof Let $G = C_n@K_{1,2}$ with $V(G) = \{u_i : 1 \le i \le n\} \cup \{v_1, v_2\}$, $E(G) = \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_nu_1, u_nv_n, u_nv_2\}$. So, $|V(G)| = n+2$ and $|E(G)| = n+2$. Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_a\}, a\ \epsilon\ N$ by $f(u_1) = 0$, $f(v_1) = l_n$, $f(v_2) = l_{n+3}$; $f(u_i) = l_{2i-3s}$, $3s-1 \le i \le 3s+1$ for $s = 1, 2, ..., \dfrac{n-4}{3}$ and $f(u_i) = l_{2i-3s}$, $3s-1 \le i \le 3s$ for $s = \dfrac{n-1}{3}$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \{f_1(u_1u_2), f_1(u_nv_1), f_1(u_nv_2), f_1(u_nu_1)\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_n) - f(v_1)|, |f(u_n) - f(v_2)|, |f(u_n) - f(v_1)|\} \\
&= \{|l_0 - l_1|, |l_{n+1} - l_n|, |l_{n+1} - l_{n+3}|, |l_{n+1} - l_0|\} \\
&= \{l_1, l_{n-1}, l_{n+2}, l_{n+1}\},
\end{aligned}$$

$$\begin{aligned}
E_2 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_iu_{i+1}) : 3s-1 \le i \le 3s\} \\
&= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \le i \le 3s\} \\
&= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(u_{n-5}) - f(u_{n-4})|, |f(u_{n-4}) - f(u_{n-3})|\} \\
&= \{|l_1 - l_3|, |l_3 - l_5|\} \bigcup \{|l_4 - l_6|, |l_6 - l_8|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{n-6} - l_{n-4}|, |l_{n-5} - l_{n-2}|\} \\
&= \{l_2, l_4\} \bigcup \{l_5, l_7\} \bigcup \cdots \bigcup \{l_{n-5}, l_{n-3}\} = \{l_2, l_4, l_5, l_7, \cdots, l_{n-5}, l_{n-3}\}
\end{aligned}$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $1 \leq s \leq \dfrac{n-4}{3}$. Let

$$\begin{aligned} E_3 &= \{f_1(u_iu_{i+1}) : i = 3s+1\} = \{|f(u_i) - f(u_{i+1})| : i = 3s+1\} \\ &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \cdots, |f(u_{n-3}) - f(u_{n-2})|\} \\ &= \{|l_{8-3} - l_{10-6}|, |l_{14-6} - l_{16-9}|, \cdots, |l_{2n-6-n+4} - l_{2n-4-n+1}|\} \\ &= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{n-2} - l_{n-3}|\} = \{l_3, l_6, \cdots, l_{n-4}\}. \end{aligned}$$

For $s = \frac{n-1}{3}$, let

$$\begin{aligned} E_4 &= \{f_1(u_iu_{i+1}) : 3s-1 \leq i \leq 3s\} \\ &= \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\ &= \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|l_{2n-4-n+1} - l_{2n-2-n+1}|, |l_{2n-2-n+1} - l_{2n-n+1}|\} \\ &= \{|l_{n-3} - l_{n-1}|, |l_{n-1} - l_{n+1}|\} = \{l_{n-2}, l_n\} \end{aligned}$$

Now, $E = \bigcup_{i=1}^{4} E_i = \{l_1, l_2, ..., l_{n+2}\}$. So, the edge labels of G are distinct. Therefore, f is a Lucas graceful labeling. Hence, $G = C_n@K_{1,2}$ is a Lucas graceful graph. □

Example 2.21 The graph $C_{10}@K_{1,2}$ admits Lucas graceful labeling shown in Fig.9.

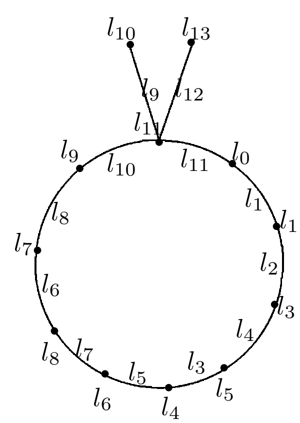

Fig.9

§3. Strong Lucas Graceful Graphs

In this section, we prove that the graphs $K_{1,n}$ and F_n admit strong Lucas graceful labeling.

Definition 3.1 *Let G be a (p,q) graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_q\}$ is said to be strong Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection on to the set $\{l_1, l_2, ..., l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 =$*

$7, l_5 = 11, \cdots,$. Then G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling.

Theorem 3.2 *The graph $K_{1,n}$ is a strong Lucas graceful graph.*

Proof Let $G = K_{1,n}$ and $V = V_1 \cup V_2$ be the bipartition of $K_{1,n}$ with $V_1 = \{u_1\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$. Then, $|V(G)| = n+1$ and $|E(G)| = n$. Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_n\}$ by $f(u_0) = l_0$, $f(u_1) = l_1, 1 \le i \le n$. We claim that the edge labels are distinct. Notice that

$$\begin{aligned} E &= \{f_1(u_0u_1) : 1 \le i \le n\} = \{f(u_0) - f(u_1) : 1 \le i \le n\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, ..., |f(u_0) - f(u_n)|\} \\ &= \{|l_0 - l_1|, |l_0 - l_2|, ..., |l_0 - l_n|\} = \{l_1, l_2, ..., l_n\} \end{aligned}$$

So, the edges of G receive the distinct labels. Therefore, f is a strong Lucas graceful labeling. Hence, K_1, n the path is a strong Lucas graceful graph. □

Example 3.3 The graph $K_{1,9}$ admits strong Lucas graceful labeling shown in Fig.10.

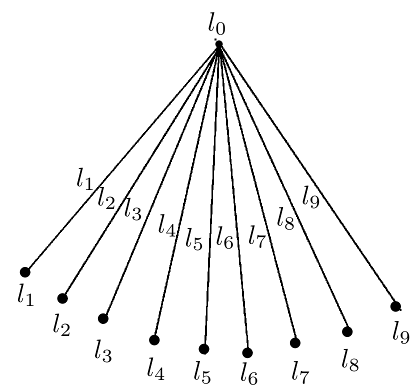

Fig.10

Definition 3.4([2]) *Let $u_1, u_2, ..., u_n, u_{n+1}$ be the vertices of a path and u_0 be a vertex which is attached with $u_1, u_2, ..., u_n, u_{n+1}$. Then the resulting graph is called Fan and is denoted by $F_n = P_n + K_1$.*

Theorem 3.5 *The graph $F_n = P_n + K_1$ is a Lucas graceful graph.*

Proof Let $G = F_n$ and $u_1, u_2, ..., u_n, u_{n+1}$ be the vertices of a path P_n with the central vertex u_0 joined with $u_1, u_2, ..., u_n, u_{n+1}$. Clearly, $|V(G)| = n+2$ and $|E(G)| = 2n+1$. Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{2n+1}\}$ by $f(u_0) = l_0$ and $f(u_i) = l_{2i-1}, 1 \le i \le n+1$. We claim that the edge labels are distinct.

Calculation shows that

$$\begin{aligned} E_1 &= \{f_1(u_iu_{i+1}) : 1 \le i \le n\} = \{|f_(u_i) - f(u_{i+1})| : 1 \le i \le n\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, ..., |f(u_n) - f(u_{n+1})|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|, ..., |l_{2n-1} - l_{2n+1}|\} = \{l_2, l_4, ..., l_{2n}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f_1(u_0u_i) : 1 \le i \le n+1\} = \{|f(u_0) - f(u_i)| : 1 \le i \le n+1\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, ..., |f(u_0) - f(u_{n+1})|\} \\ &= \{|l_0 - l_1|, |l_0 - l_3|, ..., |l_0 - l_{2n+1}|\} = \{l_1, l_3, ..., l_{2n+1}\}. \end{aligned}$$

Whence, $E = E_1 \cup E_2 = \{l_1, l_2, ..., l_{2n}, l_{2n+1}\}$. Thus the edges of F_n receive the distinct labels. Therefore, f is a Lucas graceful labeling. Consequently, $F_n = P_n + K_1$ is a Lucas graceful graph. □

Example 3.6 The graph $F_7 = P_7 + K_1$ admits Lucas graceful graph shown in Fig.11.

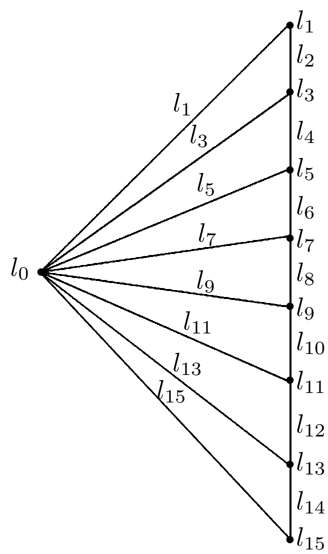

Fig.11

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