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The H-Line Signed Graph of a Signed Graph

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. Given a connected graph H of order at least 3, the *H-Line Graph* of a graph $G = (V, E)$, denoted by $HL(G)$, is a graph with the vertex set E, the edge set of G where two vertices in $HL(G)$ are adjacent if, and only if, the corresponding edges are adjacent in G and there exists a copy of H in G containing them. Analogously, for a connected graph H of order at lest 3, we define the *H-Line signed graph* $HL(S)$ of a signed graph $S = (G, \sigma)$ as a signed graph, $HL(S) = (HL(G), \sigma')$, and for any edge e_1e_2 in $HL(S)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. In this paper, we characterize signed graphs S which are H-line signed graphs and study some properties of H-line graphs as well as H-line signed graphs.

Key Words: Smarandachely k-Signed graphs, Smarandachely k-Marked graphs, Signed graphs, Balance, Switching, H-Line signed graph.

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§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [8]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed*

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graph or a *marked graph*. We say that a signed graph is *connected* if its underlying graph is connected. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [9]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ.

The following characterization of balanced signed graphs is well known.

Theorem 1.1(E. Sampathkumar [12]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S.

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [22]) or *cycle isomorphic* (see [23]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [23]):

Theorem 1.2(T. Zaslavsky [23]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. H-Line Signed Graph of a Signed Graph

The line graph $L(G)$ of a nonempty graph $G = (V, E)$ is the graph whose vertices are the edges of G and two vertices are adjacent if and only if the corresponding edges are adjacent. The triangular line graph $\mathcal{T}(G)$ of a nonempty graph was introduced by Jerret [10] as a graph whose vertices are edges of G and two vertices are adjacent if and only if corresponding edges belongs to a common triangle. Triangular graphs were introduced to model a metric space defined on the edge set of a graph. These concepts were generalized in [5] as follows: Let H be a fixed connected graph of order at least 3. For a graph G of size the *H-line graph* of G, denoted by $HL(G)$,is the graph whose vertices are the edges of G and two vertices are adjacent the corresponding edges are adjacent and belong to a copy of H.If $H \cong P_3$ then $HL(G) = L(G)$ and so H-line graph is a generalization of line graphs. Clearly, if a graph is H free, then its H-line graph is trivial.

In [10], the authors introduced the notion of triangular line graph of a graph as follows: The *triangular line graph* of a $G = (V, E)$ denoted by $\mathcal{T}(G) = (V', E')$, whose vertices are the edges of G and two vertices are adjacent the corresponding edges belongs to a triangle in G. Clearly for any graph G, $\mathcal{T}(G) = K_3L(G)$.

Behzad and Chartrand [3] introduced the notion of *line signed graph* $L(S)$ of a given signed graph S as follows: $L(S)$ is a signed graph such that $(L(S))^u \cong L(S^u)$ and an edge e_ie_j in $L(S)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [7], is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [7] (See also E. Sampathkumar et al. [13,14]). For more operations on signed graphs see [15-20].

Proposition 2.1 *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

In [21], the authors extends the notion of triangular line graphs to triangular line signed graphs. We now extend the notion of H-line graph to the realm of signed graph as follows:

Let $S = (G, \sigma)$ ba a signed graph. For any fixed connected graph H of order at least 3, the H-line signed graph of S, denoted by $HL(S)$ is the signed graph $HL(S) = (HL(G), \sigma')$ whose underlying graph is $HL(G)$ and for any edge ee' in $HL(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. Further a signed graph S is said to be H-line signed graph if there exists a signed graph S' such that $HL(S') \cong S$.

We now give a straightforward, yet interesting property of H-line signed graphs.

Theorem 2.2 *For any connected graph H of order at least 3 and for any signed graph $S = (G, \sigma)$, its H-line signed graph $HL(S)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $HL(S)$ and let the signing σ of S be treated as a marking of the vertices of $HL(S)$. Then by definition of $HL(S)$ we see that $\sigma'(e_1, e_2) = \sigma(e_1)\sigma(e_2)$, for every edge (e_1, e_2) of $HL(S)$ and hence, by Theorem 1.1, the result follows. □

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $HL(S_1) \sim HL(S_2)$.*

The following result characterizes signed graphs which are H-line signed graphs.

Theorem 2.4 *A signed graph $S = (G, \sigma)$ is a H-line signed graph for some connected graph H of order at least 3 if, and only if, S is balanced signed graph and its underlying graph G is a*

H-line graph.

Proof Suppose that S is H-line signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $HL(S') \cong S$. Hence by definition $HL(G) \cong G'$ and by Theorem 2.2, S is balanced.

Conversely, suppose that $S = (G, \sigma)$ is balanced and G is H-line graph. That is there exists a graph G' such that $HL(G') \cong G$. Since S is balanced by Theorem 1.1, there exists a marking μ of vertices of S such that for any edge $uv \in G$, $\sigma(uv) = \mu(u)\mu(v)$. Also since $G \cong HL(G')$, vertices in G are in one-to-one correspondence with the edges of G'. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e' in G' to be the marking on the corresponding vertex in G. Then clearly $HL(S') \cong S$ and so S is H-line graph. □

For any positive integer k, the k^{th} iterated H-line signed graph, $HL^k(S)$ of S is defined as follows:

$$HL^0(S) = S,\ HL^k(S) = HL(HL^{k-1}(S)).$$

Corollary 2.5 *Given a signed graph $S = (G, \sigma)$ and any positive integer k, $HL^k(S)$ is balanced, for any connected graph H of order ≥ 3.*

In [6], the authors proved the following for a graph G its H-line graph $HL(G)$ is isomorphic to G then H is a path or a cycle. Analogously we have the following.

Theorem 2.6 *If a signed graph $S = (G, \sigma)$ satisfies $S \sim HL(S)$ then S is balanced and H is a cycle or a path.*

Theorem 2.7 *For any cycle C_k on $k \geq 3$ vertices, a connected graph G on $n \geq r$ vertices satisfies $C_kL(G) \cong G$ if, and only if, $G = C_k$.*

Proof Suppose that $C_kL(G) \cong G$. Then clearly, G must be unicyclic. Since $C_kL(G) \cong G$, we observe that G must contain a cycle C_k. Next, suppose that G contains a vertex of degree ≥ 3, then the vertex corresponding to the edge not on the cycle in $C_kL(G)$ will be isolated vertex. Hence G must be a cycle C_k.

Conversely, if $G = C_k$, then clearly for any two adjacent edges in C_k belongs to a copy of C_k and so $C_kL(G) \cong L(G)$. Since the line graph of any C_k is C_k itself, we have $C_kL(G) \cong G$.□

Corollary 2.8 *For any cycle C_k on $k \geq 3$ vertices, a graph G on $n \geq r$ vertices satisfies $C_kL(G) \cong G$ if, and only if, G is 2-regular and every component of G is C_k.*

In view of the above theorem we have,

Theorem 2.9 *For any cycle C_k on $k \geq 3$ vertices, a signed graph $S = (G, \sigma)$ connected graph G on $n \geq r$ vertices satisfies $C_kL(S) \sim S$ if, and only if, $G = C_k$.*

Theorem 2.10 *For a path P_k on $k \geq 3$ vertices a connected graph G on $n \geq r$ vertices which contains a cycle of length $r > k$ satisfies $P_kL(G) \cong L(G)$ if, and only if, $G = C_n$ and $n \geq k$.*

Proof The result follows if $k = 3$, since $P_3L(G) = L(G)$. Assume that $k \geq 4$. Clearly G must contain at least k vertices. Suppose that $P_kL(G) \cong L(G)$ and G contains a cycle of length $r \geq k$. Then number of vertices in G and number of edges are equal. Hence G must be unicyclic. Since G contains a cycle of length $r > k$, then any two adjacent edges in C of G belongs to a common P_k. Hence $P_kL(G)$ also contains a cycle of length r. Next, suppose that G contains a vertex of degree ≥ 3, then the vertex corresponding to the edge not on the cycle in $P_kL(G)$ will be adjacent to two adjacent vertices forming a C_3 and so $HL(G)$ is not unicyclic. Hence G must be the cycle C_n .

Conversely, if $G = C_n$ and $n \geq k$, then clearly any two adjacent edges in C_k belongs to a copy of C_k and so $P_kL(G) \cong L(G)$. Since the line graph of C_n is C_n itself, $P_kL(G) \cong L(G)$. □

Corollary 2.11 *For any path P_k on $k \geq 3$ vertices, a graph G on $n \geq r$ vertices satisfies $P_kL(G) \cong G$ if, and only if, G is 2-regular and every component of G is C_r, for some $r \geq k$.*

Analogously, we have the following for signed graphs:

Corollary 2.12 *For any path P_k on $k \geq 3$ vertices, a signed graph $S = (G, \sigma)$ on $n \geq r$ vertices satisfies $P_kL(S) \sim S$ if, and only if, S is balanced and every component of G is C_r, for some $r \geq k$.*

In [10], the authors prove that for any graph G, $\mathcal{T}(G) \cong L(G)$ if, and only if, $G = K_n$. Analogously, we have the following:

Theorem 2.13 *A graph G of order n satisfies $K_rL(G) \cong L(G)$ for some $r \leq n$ if, and only if, $G = K_n$.*

Proof The result is trivial if $k = n$. Suppose that $K_rL(G) \cong L(G)$ and G is not complete for some $r \leq n - 1$. Then there exists at least two nonadjacent vertices u and v in G. Now for any vertex w, the edges uw and vw are adjacent and hence the corresponding vertices are adjacent. But the edges uw and vw can not be adjacent in $K_rL(G)$ since any set of r vertices containing u and v can not induce complete subgraph K_r. Whence, the condition is necessary.

For sufficiency, suppose $G = K_n$ for some $n \geq r$. Then for any two adjacent vertices in L(G), the corresponding edges adjacent edges in G belongs to some K_r. Hence they are also adjacent in $K_rL(G)$ and any two nonadjacent vertices in $L(G)$ remain nonadjacent.This completes the proof. □

Analogously, we have the following result for signed graphs:

Theorem 2.14 *A signed graph $S = (G, \sigma)$ satisfies $K_rL(S) \sim L(S)$, for some $3 \leq k \leq |V(G)|$ if, and only if, S is a balanced on a complete graph.*

§3. Triangular Line Signed Graphs and (0, 1, -1) Matrices

Matrices are very good models to represent a graph. In general given a matrix $A = (a_{ij})$ of order $m \times n$ one can associate many graphs with it (see [11]. On the other hand given any graph G we can associate many matrices such adjacency matric, incidence matrix etc (see [8]). Analogously, given a matrix with entries one can associate many signed graphs (See [11]). In this section, we give a relation between the notion of triangular line graph and some graph associated with $\{0,1\}$-matrices. Also we extend this to triangular signed graphs and some signed graphs associated with matrices whose entries are $-1, 0$, or 1.

Given a $(0,1)$-matrix A, the term graph $T(A)$ of A was defined as follows (See [2]): The vertex set of $T(A)$ consists of m row labels $r_1, r_2, ..., r_m$ and n column labels $c_1, c_2, ..., c_n$ of A and the edge set consists of the unordered pairs $r_i c_j$ for which $a_{ij} \neq 0$.

Given a $(0,1)$-matrix A of order $m \times n$, the graph $G_t(A)$ can be constructed as follows: The vertex set of $G_t(A)$ consists of non-zero entries of A and the edge set consists of distinct pairs of vertices (a_{ij}, a_{kr}) that lie in the same row (i=k) with $a_{ir} \neq 0$ or or same column(j=r) with $a_{kj} \neq 0$. The following result relates the connects the two notions the term graph and G_t graph of a given matrix A:

Theorem 3.1 *For any* $(0,1)$*-matrix A,* $G_t(A) = \mathcal{T}(T(A))$.

Let $A = (a_{ij})$ be any $m \times n$ matrix in which each entry belongs to the set $\{-1, 0, 1\}$; we shall call such a matrix a $(0, \pm 1)$-matrix. The notion of term graph of a $(0,1)$-matrix can be easily extended to term signed graph of a $(0, \pm 1)$-matrix A as follows (see [2]): The vertex set of $T(A)$ consists of m row labels $r_1, r_2, ..., r_m$ and n column labels $c_1, c_2, ..., c_n$ of A, the edge set consists of the unordered pairs $r_i c_j$ for which $a_{ij} \neq 0$ and the sign of the edge $r_i c_j$ is the sign of the nonzero entry a_{ij}.

Next, given any $(0, \pm 1)$-matrix A a *triangular matrix signed graph* $Sg_t(A)$ of A can be constructed as follows: The vertex set of $Sg_t(A)$ is consists of nonzero entries of A and edge set consists of distinct pairs of vertices (a_{ij}, a_{kr}) that lie in the same row $(i = k)$ with $a_{ir} \neq 0$ or same column $(j = r)$ with $a_{kj} \neq 0$; the sign of an edge uv in $Sg(A)$ is defined as the product of sings of the entries of A that correspond to $u = a_{ij}$ and $v = a_{kr}$.

The following is a observation whose proof follows from the definition of triangular line graph and the facts just mentioned above:

Theorem 3.2 *For any* $(0, \pm 1)$ *matrix A,* $Sg_t(A) \cong \mathcal{T}(T_g(A))$.

The *Kronecker product* or *tensor product* of two signed graphs S_1 and S_2, denoted by $S_1 \bigotimes S_2$ is defined (see [2]) as follows:The vertex set of $(S_1 \bigotimes S_2)$ is $V(S_1) \times V(S_2)$, the edge set is $E(S_1 \bigotimes S_2) := \{((u_1, v_1), (u_2, v_2)) : u_1 u_2 \in E(S_1),\ v_1 v_2 \in E(S_2)\}$ and the sign of the edge $(u_1, v_1)(u_2, v_2)$ is the product of the sign of $u_1 u_2$ in S_1 and the sign of $v_1 v_2$ in S_2. In the following result, $A(S)$ will denote the usual adjacency matrix of the given signed graph S and $A \bigotimes B$ denotes the standard tensor product of the given matrices A and B.

Theorem 3.3(M. Acharya [2]) *For any two signed graphs* $S_1 and S_2$*,* $A(S_1 \bigotimes S_2) = A(S_1) \bigotimes A(S_2)$.

Theorem 3.4 *For any signed graph S,* $T(A(S)) = S \bigotimes K_2^+$*, where* K_2^+ *denotes the complete graph* K_2 *with its only edge treated as being positive.*

The *adjacency signed graph* $\eth(S)$ of a given signed graph S is the matrix signed graph $Sg(A(S))$ of the adjacency matrix $A(S)$ of S [2].

Theorem 3.5(M. Acharya [2]) *For any signed graph S,* $\eth(S) = L(S \bigotimes K_2^+)$.

Analogously we define *triangular adjacency signed graph* of $A(S)$, the adjacency matrix of S denoted by $\eth_t(S)$ as the signed graph $Sg_t(A(S))$. We have the following result.

Theorem 3.6 *For any signed graph S,* $\eth_t(S) = \mathcal{T}(S \bigotimes K_2^+)$.

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