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An holomorphic study of Smarandache automorphic and cross inverse property loops

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Abstract By studying the holomorphic structure of automorphic inverse property quasigroups and loops[AIPQ and (AIPL)] and cross inverse property quasigroups and loops[CIPQ and (CIPL)], it is established that the holomorph of a loop is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop if and only if its Smarandache automorphism group is trivial and the loop is itself is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop.

Keywords Smarandache loop, holomorph of loop, automorphic inverse property loop (AIPL), cross inverse property loop(CIPL), K-loop, Bruck-loop, Kikkawa-loop.

§1. Introduction

1.1 Quasigroups and loops

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of x respectively. Now, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. To every loop $(L, \cdot)$ with automorphism group $AUM(L, \cdot)$, there corresponds another loop. Let the set $H = (L, \cdot) \times AUM(L, \cdot)$. If we define '$\circ$' on H such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H$, then $H(L, \cdot) = (H, \circ)$ is a loop as shown in Bruck [7] and is called the Holomorph of $(L, \cdot)$. A loop(quasigroup) is a weak inverse property loop (quasigroup)[WIPL(WIPQ)] if and only if it obeys the identity

$$x(yx)^\rho = y^\rho \qquad \text{or} \qquad (xy)^\lambda x = y^\lambda.$$

A loop(quasigroup) is a cross inverse property loop(quasigroup)[CIPL(CIPQ)] if and only if it obeys the identity

$$xy \cdot x^\rho = y \qquad \text{or} \qquad x \cdot yx^\rho = y \qquad \text{or} \qquad x^\lambda \cdot (yx) = y \qquad \text{or} \qquad x^\lambda y \cdot x = y.$$

A loop(quasigroup) is an automorphic inverse property loop(quasigroup)[AIPL(AIPQ)] if and only if it obeys the identity

$$(xy)^\rho = x^\rho y^\rho \ or \ (xy)^\lambda = x^\lambda y^\lambda$$

Consider $(G, \cdot)$ and $(H, \circ)$ being two distinct groupoids(quasigroups, loops). Let A, B and C be three distinct non-equal bijective mappings, that maps G onto H. The triple $\alpha = (A, B, C)$ is called an isotopism of $(G, \cdot)$ onto $(H, \circ)$ if and only if

$$xA \circ yB = (x \cdot y)C \ \forall \ x, y \in G.$$

The set $SYM(G, \cdot) = SYM(G)$ of all bijections in a groupoid $(G, \cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(G, \cdot)$. If $(G, \cdot) = (H, \circ)$, then the triple $\alpha = (A, B, C)$ of bijections on $(G, \cdot)$ is called an autotopism of the groupoid(quasigroup, loop) $(G, \cdot)$. Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of the groupoid(quasigroup, loop) $(G, \cdot)$. Such bijections form a group $AUM(G, \cdot)$ called the automorphism group of $(G, \cdot)$.

The left nucleus of L denoted by $N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall \ x, y \in L\}$. The right nucleus of L denoted by $N_\rho(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall \ x, y \in L\}$. The middle nucleus of L denoted by $N_\mu(L, \cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall \ x, y \in L\}$. The nucleus of L denoted by $N(L, \cdot) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot)$. The centrum of L denoted by $C(L, \cdot) = \{a \in L : ax = xa \ \forall \ x \in L\}$. The center of L denoted by $Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot)$.

As observed by Osborn [22], a loop is a WIPL and an AIPL if and only if it is a CIPL. The past efforts of Artzy [2], [3], [4] and [5], Belousov and Tzurkan [6] and recent studies of Keedwell [17], Keedwell and Shcherbacov [18], [19]and [20] are of great significance in the study of WIPLs, AIPLs, CIPQs and CIPLs, their generalizations(i.e m-inverse loops and quasigroups, (r,s,t)-inverse quasigroups) and applications to cryptography. For more on loops and their properties, readers should check [8], [10], [12], [13], [27] and [24].

Interestingly, Adeniran [1] and Robinson [25], Oyebo and Adeniran [23], Chiboka and Solarin [11], Bruck [7], Bruck and Paige [9], Robinson [26], Huthnance [14] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops, Osborn loops and Bruck loops. Huthnance [14] showed that if $(L, \cdot)$ is a loop with holomorph $(H, \circ)$, $(L, \cdot)$ is a WIPL if and only if $(H, \circ)$ is a WIPL. The holomorphs of an AIPL and a CIPL are yet to be studied.

For the definitions of inverse property loop (IPL), Bol loop and A-loop readers can check earlier references on loop theory.

Here, a K-loop is an A-loop with the AIP, a Bruck loop is a Bol loop with the AIP and a Kikkawa loop is an A-loop with the IP and AIP.

1.2 Smarandache quasigroups and loops

The study of Smarandache loops was initiated by W. B. Vasantha Kandasamy in 2002. In her book [27], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. In [16], the present author defined a Smarandache quasigroup (S-quasigroup) to be a quasigroup with at least a non-trivial associative subquasigroup called a Smarandache subsemigroup (S-subsemigroup). Examples of Smarandache quasigroups are given in Muktibodh [21]. In her book, she introduced over 75 Smarandache concepts on loops. In her first paper [28], on the study of Smarandache notions in algebraic structures, she introduced Smarandache : left(right) alternative loops, Bol loops, Moufang loops, and Bruck loops. But in [15], the present author introduced Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops.

A loop is called a Smarandache A-loop(SAL) if it has at least a non-trivial subloop that is a A-loop.

A loop is called a Smarandache K-loop(SKL) if it has at least a non-trivial subloop that is a K-loop.

A loop is called a Smarandache Bruck-loop(SBRL) if it has at least a non-trivial subloop that is a Bruck-loop.

A loop is called a Smarandache Kikkawa-loop(SKWL) if it has at least a non-trivial subloop that is a Kikkawa-loop.

If L is a S-groupoid with a S-subsemigroup H, then the set $SSYM(L,\cdot) = SSYM(L)$ of all bijections A in L such that $A : H \to H$ forms a group called the Smarandache permutation(symmetric) group of the S-groupoid. In fact, $SSYM(L) \leq SYM(L)$.

The left Smarandache nucleus of L denoted by $SN_\lambda(L,\cdot) = N_\lambda(L,\cdot) \cap H$. The right Smarandache nucleus of L denoted by $SN_\rho(L,\cdot) = N_\rho(L,\cdot) \cap H$. The middle Smarandache nucleus of L denoted by $SN_\mu(L,\cdot) = N_\mu(L,\cdot) \cap H$. The Smarandache nucleus of L denoted by $SN(L,\cdot) = N(L,\cdot) \cap H$. The Smarandache centrum of L denoted by $SC(L,\cdot) = C(L,\cdot) \cap H$. The Smarandache center of L denoted by $SZ(L,\cdot) = Z(L,\cdot) \cap H$.

Definition 1.1. Let $(L,\cdot)$ and $(G,\circ)$ be two distinct groupoids that are isotopic under a triple (U,V,W). Now, if $(L,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $A : L' \to G'$, where $A \in \{U,V,W\}$, then the isotopism $(U,V,W) : (L,\cdot) \to (G,\circ)$ is called a Smarandache isotopism(S-isotopism).

Thus, if $U = V = W$, then U is called a Smarandache isomorphism, hence we write $(L,\cdot) \succsim (G,\circ)$.

But if $(L,\cdot) = (G,\circ)$, then the autotopism (U,V,W) is called a Smarandache autotopism (S-autotopism) and they form a group $SAUT(L,\cdot)$ which will be called the Smarandache autotopism group of $(L,\cdot)$. Observe that $SAUT(L,\cdot) \leq AUT(L,\cdot)$. Furthermore, if $U = V = W$, then U is called a Smarandache automorphism of $(L,\cdot)$. Such Smarandache permutations form a group $SAUM(L,\cdot)$ called the Smarandache automorphism group(SAG) of $(L,\cdot)$.

Let L be a S-quasigroup with a S-subgroup G. Now, set $H_S = (G,\cdot) \times SAUM(L,\cdot)$. If we define '$\circ$' on H_S such that $(\alpha,x) \circ (\beta,y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha,x),(\beta,y) \in H_S$, then $H_S(L,\cdot) = (H_S,\circ)$ is a quasigroup.

If in L, $s^\lambda \cdot s\alpha \in SN(L)$ or $s\alpha \cdot s^\rho \in SN(L)$ $\forall s \in G$ and $\alpha \in SAUM(L,\cdot)$, $(H_S,\circ)$ is called a Smarandache Nuclear-holomorph of L, if $s^\lambda \cdot s\alpha \in SC(L)$ or $s\alpha \cdot s^\rho \in SC(L)$ $\forall$ $s \in G$ and $\alpha \in SAUM(L,\cdot)$, $(H_S,\circ)$ is called a Smarandache Centrum-holomorph of L hence a Smarandache Central-holomorph if $s^\lambda \cdot s\alpha \in SZ(L)$ or $s\alpha \cdot s^\rho \in SZ(L)$ $\forall$ $s \in G$ and $\alpha \in SAUM(L,\cdot)$.

The aim of the present study is to investigate the holomorphic structure of Smarandache AIPLs and CIPLs(SCIPLs and SAIPLs) and use the results to draw conclusions for Smarandache K-loops(SKLs), Smarandache Bruck-loops(SBRLs) and Smarandache Kikkawa-loops (SKWLs). This is done as follows.

1. The holomorphic structure of AIPQs(AIPLs) and CIPQs(CIPLs) are investigated. Necessary and sufficient conditions for the holomorph of a quasigroup(loop) to be an AIPQ(AIPL) or CIPQ(CIPL) are established. It is shown that if the holomorph of a quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL), then the holomorph is isomorphic to the quasigroup(loop). Hence, the holomorph of a quasigroup(loop) is an AIPQ(AIPL) or CIPQ(CIPL) if and only if its automorphism group is trivial and the quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL). Furthermore, it is discovered that if the holomorph of a quasigroup(loop) is a CIPQ(CIPL), then the quasigroup(loop) is a flexible unipotent CIPQ(flexible CIPL of exponent 2).

2. The holomorph of a loop is shown to be a SAIPL, SCIPL, SKL, SBRL or SKWL respectively if and only its SAG is trivial and the loop is a SAIPL, SCIPL, SKL, SBRL, SKWL respectively.

§2. Main results

Theorem 2.1. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is an AIPQ(AIPL) if and only if

1. $AUM(L)$ is an abelian group,
2. $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ and
3. L is a AIPQ(AIPL).

Proof. A quasigroup(loop) is an automorphic inverse property loop(AIPL) if and only if it obeys the AIP identity.

Using either of the definitions of an AIPQ(AIPL), it can be shown that $H(L)$ is a AIPQ(AIPL) if and only if $AUM(L)$ is an abelian group and $(\beta^{-1}J_\rho, \alpha J_\rho, J_\rho) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$. L is isomorphic to a subquasigroup(subloop) of $H(L)$, so L is a AIPQ(AIPL) which implies $(J_\rho, J_\rho, J_\rho) \in AUT(L)$. So, $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.

Corollary 2.1. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is a CIPQ(CIPL) if and only if

1. $AUM(L)$ is an abelian group,
2. $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ and
3. L is a CIPQ(CIPL).

Proof. A quasigroup(loop) is a CIPQ(CIPL) if and only if it is a WIPQ(WIPL) and an AIPQ(AIPL). L is a WIPQ(WIPL) if and only if $H(L)$ is a WIPQ(WIPL).

If $H(L)$ is a CIPQ(CIPL), then $H(L)$ is both a WIPQ(WIPL) and a AIPQ(AIPL) which implies 1., 2., and 3. of Theorem 2.1. Hence, L is a CIPQ(CIPL). The converse follows by just doing the reverse.

Corollary 2.2. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. If $H(L)$ is an AIPQ(AIPL) or CIPQ(CIPL), then $H(L) \cong L$.

Proof. By 2. of Theorem 2.1, $(\beta^{-1}, \alpha, I) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$ implies $x\beta^{-1} \cdot y\alpha = x \cdot y$ which means $\alpha = \beta = I$ by substituting $x = e$ and $y = e$. Thus, $AUM(L) = \{I\}$ and so $H(L) \cong L$.

Theorem 2.2. The holomorph of a quasigroup(loop) L is a AIPQ(AIPL) or CIPQ(CIPL) if and only if $AUM(L) = \{I\}$ and L is a AIPQ(AIPL) or CIPQ(CIPL).

Proof. This is established using Theorem 2.1, Corollary 2.1 and Corollary 2.2.

Theorem 2.3. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. $H(L)$ is a CIPQ(CIPL) if and only if $AUM(L)$ is an abelian group and any of the following is true for all $x, y \in L$ and $\alpha, \beta \in AUM(L)$.

1. $(x\beta \cdot y)x^\rho = y\alpha$.

2. $x\beta \cdot yx^\rho = y\alpha$.

3. $(x^\lambda\alpha^{-1}\beta\alpha \cdot y\alpha) \cdot x = y$.

4. $x^\lambda\alpha^{-1}\beta\alpha \cdot (y\alpha \cdot x) = y$.

Proof. This is achieved by simply using the four equivalent identities that define a CIPQ(CIPL):

Corollary 2.3. Let $(L, \cdot)$ be a quasigroups(loop) with holomorph $H(L)$. If $H(L)$ is a CIPQ(CIPL) then, the following are equivalent to each other

1. $(\beta^{-1}J_\rho, \alpha J_\rho, J_\rho) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.

2. $(\beta^{-1}J_\lambda, \alpha J_\lambda, J_\lambda) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.

3. $(x\beta \cdot y)x^\rho = y\alpha$.

4. $x\beta \cdot yx^\rho = y\alpha$.

5. $(x^\lambda\alpha^{-1}\beta\alpha \cdot y\alpha) \cdot x = y$.

6. $x^\lambda\alpha^{-1}\beta\alpha \cdot (y\alpha \cdot x) = y$.

Hence, $(\beta, \alpha, I), (\alpha, \beta, I), (\beta, I, \alpha), (I, \alpha, \beta) \in AUT(L)\ \forall\ \alpha, \beta \in AUM(L)$.

Proof. The equivalence of the six conditions follows from Theorem 2.3 and the proof of Theorem 2.1. The last part is simple.

Corollary 2.4. Let $(L, \cdot)$ be a quasigroup(loop) with holomorph $H(L)$. If $H(L)$ is a CIPQ(CIPL) then, L is a flexible unipotent CIPQ(flexible CIPL of exponent 2).

Proof. It is observed that $J_\rho = J_\lambda = I$. Hence, the conclusion follows.

Remark. The holomorphic structure of loops such as extra loop, Bol-loop, C-loop, CC-loop and A-loop have been found to be characterized by some special types of automorphisms such as

1. Nuclear automorphism(in the case of Bol-,CC- and extra loops),

2. central automorphism(in the case of central and A-loops).

By Theorem 2.1 and Corollary 2.1, the holomorphic structure of AIPLs and CIPLs is characterized by commutative automorphisms.

Theorem 2.4. The holomorph $H(L)$ of a quasigroup(loop) L is a Smarandache AIPQ(AIPL) or CIPQ(CIPL) if and only if $SAUM(L) = \{I\}$ and L is a Smarandache AIPQ(AIPL) or CIPQ(CIPL).

Proof. Let L be a quasigroup with holomorph $H(L)$. If $H(L)$ is a SAIPQ(SCIPQ), then there exists a S-subquasigroup $H_S(L) \subset H(L)$ such that $H_S(L)$ is a AIPQ(CIPQ). Let $H_S(L) = G \times SAUM(L)$ where G is the S-subquasigroup of L. From Theorem 2.2, it can be seen that $H_S(L)$ is a AIPQ(CIPQ) if and only if $SAUM(L) = \{I\}$ and G is a AIPQ(CIPQ). So the conclusion follows.

Corollary 2.5. The holomorph $H(L)$ of a loop L is a SKL or SBRL or SKWL if and only if $SAUM(L) = \{I\}$ and L is a SKL or SBRL or SKWL.

Proof. Let L be a loop with holomorph $H(L)$. Consider the subloop $H_S(L)$ of $H(L)$ such that $H_S(L) = G \times SAUM(L)$ where G is the subloop of L.

1. Recall that by [Theorem 5.3, [9]], $H_S(L)$ is an A-loop if and only if it is a Smarandache Central-holomorph of L and G is an A-loop. Combing this fact with Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop L is a SKL if and only if $SAUM(L) = \{I\}$ and L is a SKL.

2. Recall that by [25] and [1], $H_S(L)$ is a Bol loop if and only if it is a Smarandache Nuclear-holomorph of L and G is a Bol-loop. Combing this fact with Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop L is a SBRL if and only if $SAUM(L) = \{I\}$ and L is a SBRL.

3. Following the first reason in 1., and using Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop L is a SKWL if and only if $SAUM(L) = \{I\}$ and L is a SKWL.

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