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On the hybrid mean value of the Smarandache kn-digital sequence and Smarandache function [1]

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Abstract The main purpose of this paper is using the elementary method to study the hybrid mean value properties of the Smarandache kn-digital sequence and Smarandache function, and give an interesting asymptotic formula for it.

Keywords Smarandache kn-digital sequence, Smarandache function, hybrid mean value, as -ymptotic formula, elementary method.

§1. Introduction

For any positive integer k, the famous Smarandache kn-digital sequence $a(k,n)$ is defined as all positive integers which can be partitioned into two groups such that the second part is k times bigger than the first. For example, Smarandache $2n$ and $3n$ digital sequences $a(2,n)$ and $a(3,n)$ are defined as $\{a(2,n)\} = \{12, 24, 36, 48, 510, 612, 714, 816, \cdots\}$ and $\{a(3,n)\} = \{13, 26, 39, 412, 515, 618, 721, 824, \cdots\}$.

Recently, Professor Gou Su told me that she studied the hybrid mean value properties of the Smarandache kn-digital sequence and the divisor sum function $\sigma(n)$, and proved that the asymptotic formula

$$\sum_{n\le x} \frac{\sigma(n)}{a(k,n)} = \frac{3\pi^2}{k\cdot 20\cdot \ln 10}\cdot \ln x + O(1)$$

holds for all integers $1 \le k \le 9$.

When I read professor Gou Su's work, I found that the method is very new, and the results are also interesting. This paper as a note of Gou Su's work, we consider the hybrid mean value properties of the Smarandache kn-digital sequence and Smarandache function $S(n)$, which is defined as the smallest positive integer m such that $n|m!$. That is, $S(n) = \min\{m : n|m!,\ m \in N\}$. In this paper, we will use the elementary and analytic methods to study a similar problem, and prove a new conclusion. That is, we shall prove the following:

Theorem. Let $1 \le k \le 9$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} \frac{S(n)}{a(k,n)} = \frac{3\pi^2}{k\cdot 20}\cdot \ln\ln x + O(1).$$

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§2. Proof of the theorem

In this section, we shall use the elementary and combinational methods to complete the proof of our theorem. First we need following:

Lemma. For any real number $x > 1$, we have

$$\sum_{n\le x}\frac{S(n)}{n} = \frac{\pi^2}{6}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Proof. For any real number $x > 2$, from [4] we have the asymptotic formula

$$\sum_{n\le x} S(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \tag{1}$$

Then from Euler summation formula (see theorem 3.1 of [3]) we can deduce that

$$\begin{aligned}\sum_{1<n\le x}\frac{S(n)}{n} &= \frac{1}{x}\left(\frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right)\right) + \int_1^x \left(\frac{\pi^2}{12}\cdot\frac{t^2}{\ln t} + O\left(\frac{t^2}{\ln^2 t}\right)\frac{1}{t^2}\right) dt \\ &= \frac{\pi^2}{12}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right) + \frac{\pi^2}{12}\cdot\frac{x}{\ln x} + \frac{13\pi^2}{12}\int_1^x \frac{1}{\ln^2 t}\, dt \\ &= \frac{\pi^2}{6}\cdot\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).\end{aligned}$$

This proves our Lemma.

Now we take $k = 2$ (or $k = 4$), then for any real number $x > 1$, there exists a positive integer M such that

$$5\cdot 10^M \le x < 5\cdot 10^{M+1},$$

then we can deduce that

$$M = \frac{1}{\ln 10}\cdot \ln x + O(1). \tag{2}$$

So from the definition of $a(2, n)$ we have

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(2,n)} &= \sum_{n=1}^{4}\frac{S(n)}{a(2,n)} + \sum_{n=5}^{49}\frac{S(n)}{a(2,n)} + \sum_{n=50}^{499}\frac{S(n)}{a(2,n)} + \cdots + \sum_{n=5\cdot 10^{M-1}}^{5\cdot 10^M-1}\frac{S(n)}{a(2,n)} \\ &\quad + \sum_{5\cdot 10^M\le n\le x}\frac{S(n)}{a(2,n)} \\ &= \sum_{n=1}^{4}\frac{S(n)}{n\cdot(10+2)} + \sum_{n=5}^{49}\frac{S(n)}{n\cdot(10^2+2)} + \sum_{n=50}^{499}\frac{S(n)}{n\cdot(10^3+2)} + \cdots \\ &\quad + \sum_{n=5\cdot 10^{M-1}}^{5\cdot 10^M-1}\frac{S(n)}{n\cdot(10^{M+1}+2)} + \sum_{5\cdot 10^M\le n\le x}\frac{S(n)}{n\cdot(10^{M+2}+2)}\end{aligned} \tag{3}$$

and

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(4,n)} &= \sum_{n=1}^{2}\frac{S(n)}{a(4,n)}+\sum_{n=3}^{24}\frac{S(n)}{a(4,n)}+\sum_{n=25}^{249}\frac{S(n)}{a(4,n)}+\cdots+\sum_{n=\frac{1}{4}\cdot 10^{M-1}}^{\frac{1}{4}\cdot 10^{M}-1}\frac{S(n)}{a(4,n)}\\ &\quad+\sum_{\frac{1}{4}\cdot 10^{M}\le n\le x}\frac{S(n)}{a(4,n)}\\ &= \sum_{n=1}^{2}\frac{S(n)}{n\cdot(10+4)}+\sum_{n=3}^{24}\frac{S(n)}{n\cdot(10^2+4)}+\sum_{n=25}^{249}\frac{S(n)}{n\cdot(10^3+4)}+\cdots\\ &\quad+\sum_{n=\frac{1}{4}\cdot 10^{M-1}}^{\frac{1}{4}\cdot 10^{M}-1}\frac{S(n)}{n\cdot(10^M+4)}+\sum_{\frac{1}{4}\cdot 10^{M}\le n\le x}\frac{S(n)}{n\cdot(10^{M+1}+4)}. \qquad (4)\end{aligned}$$

Then from (2), (3) and Lemma we may immediately deduce

$$\begin{aligned}\sum_{n=5\cdot 10^{k-1}}^{5\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k+1}+2)} &= \sum_{n\le 5\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k+1}+2)}-\sum_{n\le 5\cdot 10^{k-1}}\frac{S(n)}{n\cdot(10^{k+1}+2)}\\ &= \frac{\pi^2}{6}\cdot\frac{5\cdot 10^k-5\cdot 10^{k-1}}{10^{k+1}+2}\cdot\frac{1}{\ln(5\cdot 10^k)}+O\left(\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{40}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right) \qquad (5)\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=\frac{1}{4}\cdot 10^{k-1}}^{\frac{1}{4}\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k}+4)} &= \sum_{n\le \frac{1}{4}\cdot 10^{k}-1}\frac{S(n)}{n\cdot(10^{k}+4)}-\sum_{n\le \frac{1}{4}\cdot 10^{k-1}}\frac{S(n)}{n\cdot(10^{k}+4)}\\ &= \frac{\pi^2}{6}\cdot\frac{\frac{1}{4}\cdot 10^k-\frac{1}{4}\cdot 10^{k-1}}{10^{k}+4}\cdot\frac{1}{\ln(\frac{1}{4}\cdot 10^k)}+O\left(\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{80}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right). \qquad (6)\end{aligned}$$

Noting that the identity $\sum_{n=1}^{\infty}\frac{1}{n^2}=\pi^2/6$ and the asymptotic formula

$$\sum_{1\le k\le M}\frac{1}{k}=\ln M+\gamma+O\left(\frac{1}{M}\right),$$

where γ is the Euler constant.

From (2), (3) and (5) we have

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(2,n)} &= \sum_{n=1}^{4}\frac{S(n)}{a(2,n)}+\sum_{n=5}^{49}\frac{S(n)}{a(2,n)}+\sum_{n=50}^{499}\frac{S(n)}{a(2,n)}+\cdots+\sum_{n=5\cdot 10^{M-1}}^{5\cdot 10^{M}-1}\frac{S(n)}{a(2,n)}\\ &\quad+\sum_{5\cdot 10^{M}\le n\le x}\frac{S(n)}{a(2,n)}\\ &= \sum_{k=1}^{M}\frac{3\pi^2}{40}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{40}\ln\ln x+O(1).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{1\le n\le x}\frac{S(n)}{a(4,n)} &= \sum_{n=1}^{2}\frac{S(n)}{a(4,n)}+\sum_{n=3}^{24}\frac{S(n)}{a(4,n)}+\sum_{n=25}^{249}\frac{S(n)}{a(4,n)}+\cdots+\sum_{n=\frac{1}{4}\cdot 10^{M-1}}^{\frac{1}{4}\cdot 10^{M}-1}\frac{S(n)}{a(4,n)}\\ &\quad+\sum_{\frac{1}{4}\cdot 10^{M}\le n\le x}\frac{S(n)}{a(4,n)}\\ &= \sum_{k=1}^{M}\frac{3\pi^2}{80}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right)\\ &= \frac{3\pi^2}{80}\ln\ln x+O(1).\end{aligned}$$

For using the same methods, we can also prove that the theorem holds for all integers $k=1,3,5,6,7,8,9$. This completes the proof of our theorem.

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