HYBRID MEAN VALUE ON SOME SMARANDACHE-TYPE MULTIPLICATIVE FUNCTIONS AND THE MANGOLDT FUNCTION*

Liu Huaning
Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
hnliu@nwu.edu.cn

Gao Jing
School of Science, Xi'an Jiaotong University, Xi'an, Shaanxi, P.R.China

Abstract In this paper, we study the hybrid mean value of some Smarandache-type multiplicative functions and the Mangoldt function, and give two asymptotic formulae.

Keywords: Smarandache-type multiplicative functions; Mangoldt function; Hybrid mean value.

§1. Introduction

In [1], Henry Bottomley considered eleven particular families of interrelated multiplicative functions, which are listed in Smarandache's problems.

It might be interesting to discuss the mean value of these functions on $\{p^\alpha\}$, since they are multiplicative. In this paper we study the hybrid mean value of some Smarandache-type multiplicative functions and the Mangoldt function. One is $C_m(n)$, which is defined as the m-th root of largest m-th power dividing n. The other function $J_m(n)$ is denoted as m-th root of smallest m-th power divisible by n. We will give two asymptotic formulae on these two functions. That is, we shall prove the following:

Theorem 1. *For any integer $m \geq 3$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} \Lambda(n) C_m(n) = x + O\left(\frac{x}{\log x}\right),$$

where $\Lambda(n)$ is the Mangoldt function.

*This work is supported by the N.S.F.(10271093) and the P.S.F. of P.R.China.

Theorem 2. *For any integer $m \geq 2$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} \Lambda(n) J_m(n) = x^2 + O\left(\frac{x^2}{\log x}\right).$$

Using our methods one should be able to get some similar mean value formulae. We are hoping to see more papers.

§2. Proof of the theorems

Now we prove the theorems. Noting that

$$C_m(p^\alpha) = p^k, \quad \text{if } mk \leq \alpha < m(k+1) \tag{1}$$

and

$$C_m(p^\alpha) \leq p^{\frac{\alpha}{m}}, \tag{2}$$

then we have

$$\sum_{n \leq x} \Lambda(n) C_m(n) = \sum_{p^\alpha \leq x} \log p C_m(p^\alpha) = \sum_{p \leq x} \log p C_m(p) + \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \log p C_m(p^\alpha). \tag{3}$$

Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime;} \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\sum_{n \leq x} a(n) = \pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

By Abel's identity and (1) we have

$$\begin{aligned} \sum_{p \leq x} \log p C_m(p) &= \sum_{p \leq x} \log p \sum_{n \leq x} a(n) \log n = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} \mathrm{d}t \\ &= x + O\left(\frac{x}{\log x}\right) + O\left(\int_2^x \frac{1}{\log t} \mathrm{d}t\right) = x + O\left(\frac{x}{\log x}\right). \end{aligned} \tag{4}$$

From (2) we also have

$$\begin{aligned} \sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} \log p C_m(p^\alpha) &= \sum_{2 \leq \alpha \leq \frac{\log x}{\log 2}} \sum_{p \leq x^{\frac{1}{\alpha}}} \log p C_m(p^\alpha) \leq \sum_{2 \leq \alpha \leq \frac{\log x}{\log 2}} \sum_{p \leq x^{\frac{1}{\alpha}}} \log p \cdot p^{\frac{\alpha}{m}} \\ &\ll \sum_{2 \leq \alpha \leq \frac{\log x}{\log 2}} x^{\frac{1}{m} + \frac{1}{\alpha}} \ll x^{\frac{1}{m} + \frac{1}{2}}. \end{aligned} \tag{5}$$

Therefore for any integer $m \geq 3$ and real number $x \geq 1$, from (3), (4) and (5) we have

$$\sum_{n \leq x} \Lambda(n) C_m(n) = x + O\left(\frac{x}{\log x}\right).$$

This proves Theorem 1.

On the other hand, noting that

$$J_m(p^\alpha) = p^{k+1}, \quad \text{if } mk < \alpha \leq m(k+1)$$

and

$$J_m(p^\alpha) \leq p^{\frac{\alpha}{m}+1},$$

then using the methods of proving Theorem 1 we can easily get Theorem 2.

References

[1] Henry Bottomley, Some Smarandache-Type Multiplicative Functions, Smarandache Notions Journal, **13** (2002), 134-135.