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# The hybrid mean value of the Smarandache function and the Mangoldt function

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**Abstract** For any positive integer $n$, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer $m$ such that $n|m!$. The main purpose of this paper is using the elementary methods to study the hybrid mean value of the Smarandache function $S(n)$ and the Mangoldt function $\Lambda(n)$, and prove an interesting hybrid mean value formula for $S(n)\Lambda(n)$.
**Keywords** F. Smarandache function, Mangoldt function, hybrid mean value, asymptotic formula

## §1. Introduction

For any positive integer $n$, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer $m$ such that $n|m!$. That is, $S(n) = \min\{m : \ n|m!, \ m \in N\}$. From the definition of $S(n)$ one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the factorization of $n$ into prime powers, then $S(n) = \max_{1 \le i \le k} \{S(p_i^{\alpha_i})\}$. From this formula we can easily get $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $S(13) = 13$, $S(14) = 7$, $S(15) = 5$, $S(16) = 6$, $\cdots$. About the elementary properties of $S(n)$, many people had studied it, and obtained some important results. For example, Wang Yongxing [2] studied the mean value properties of $S(n)$, and obtained that:

$$\sum_{n \le x} S(n) = \frac{\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Lu Yaming [3] studied the positive integer solutions of an equation involving the function $S(n)$, and proved that for any positive integer $k \ge 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinity positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [4] obtained some inequalities involving the F.Smarandache function. That is, he proved that for any positive integer $k \ge 2$, there exists infinite positive integer $(m_1, m_2, \cdots, m_k)$ such that the inequalities

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

$(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

On the other hand, Dr. Xu Zhefeng [5] proved: Let $P(n)$ denotes the largest prime divisor of $n$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

The main purpose of this paper is using the elementary methods to study the hybrid mean value of the Smarandache function $S(n)$ and the Mangoldt function $\Lambda(n)$, which defined as follows:

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^\alpha,\ p \text{ be a prime, } \alpha \text{ be any positive integer;} \\ 0, & \text{otherwise.} \end{cases}$$

and prove a sharper mean value formula for $\Lambda(n)S(n)$. That is, we shall prove the following conclusion:

**Theorem.** Let $k$ be any fixed positive integer. Then for any real number $x > 1$, we have

$$\sum_{n\le x} \Lambda(n)S(n) = x^2 \cdot \sum_{i=0}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where $c_i$ $(i = 0,\ 1,\ 2,\ \cdots,\ k)$ are constants, and $c_0 = 1$.

## §2. Proof of the theorem

In this section, we shall complete the proof of the theorem. In fact from the definition of $\Lambda(n)$ we have

$$\begin{aligned} \sum_{n\le x} \Lambda(n)S(n) &= \sum_{\alpha\le \frac{\ln x}{\ln 2}} \sum_{p\le x^{\frac{1}{\alpha}}} \Lambda(p^\alpha)S(p^\alpha) = \sum_{\alpha\le \frac{\ln x}{\ln 2}} \sum_{p\le x^{\frac{1}{\alpha}}} S(p^\alpha)\ln p \\ &= \sum_{p\le x} p\cdot \ln p + \sum_{2\le\alpha\le \frac{\ln x}{\ln 2}} \sum_{p\le x^{\frac{1}{\alpha}}} S(p^\alpha)\ln p. \qquad (1) \end{aligned}$$

For any positive integer $k$, from the prime theorem we know that

$$\pi(x) = \sum_{p\le x} 1 = x \cdot \sum_{i=1}^{k} \frac{a_i}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right), \qquad (2)$$

where $a_i$ ( $i = 1,\ 2, \cdots,\ k$) are constants, and $a_1 = 1$.

From the Abel's identity (see [6] Theorem 4.2) and (2) we have

$$\sum_{p\le x} p\cdot\ln p = \pi(x)\cdot x\cdot \ln x - \int_2^x \pi(y)(\ln y+1)dy$$

$$= x\ln x\cdot x\cdot\left(\sum_{i=1}^{k}\frac{a_i}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right)\right)-\int_2^x\left(\sum_{i=1}^{k}\frac{a_i}{\ln^i y}+O\left(\frac{y}{\ln^{k+1}y}\right)\right)(\ln y+1)dy$$

$$= x^2\cdot\sum_{i=0}^{k}\frac{c_i}{\ln^i x}+O\left(\frac{x}{\ln^{k+1}x}\right), \tag{3}$$

where $c_i$ ( $i=0,\ 1,\ 2,\cdots,\ k$) are constants, and $c_0=1$.

On the other hand, applying the estimate

$$S(p^\alpha)\ll \alpha\cdot\ln p,$$

we have

$$\sum_{2\le\alpha\le\frac{\ln x}{\ln 2}}\ \sum_{p\le x^{\frac{1}{\alpha}}} S(p^\alpha)\ln p \ll \sum_{2\le\alpha\le\frac{\ln x}{\ln 2}}\ \sum_{p\le x^{\frac{1}{2}}} \alpha\cdot p\cdot\ln p\ll x\cdot\ln^2 x. \tag{4}$$

Combining (1)-(4) we have

$$\sum_{n\le x}\Lambda(n)S(n)=x^2\cdot\sum_{i=0}^{k}\frac{c_i\cdot x^2}{\ln^i x}+O\left(\frac{x^2}{\ln^{k+1}x}\right),$$

where $c_i$ ($i=0,\ 1,\ 2,\ \cdots,\ k$) are constants, and $c_0=1$.

This completes the proof of the theorem.

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