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Ideal Graph of a Graph

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Abstract: In this paper, we introduce ideal graph of a graph and study some of its properties. We characterize connectedness, isomorphism of graphs and coloring property of a graph using ideal graph. Also, we give an upper bound for chromatic number of a graph.

Key Words: Graph, Smarandachely ideal graph, ideal graph, isomorphism.

AMS(2010): 05C38, 05C60

§1. Introduction

Graphs considered here are finite, simple and undirected. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. Terms not defined here are used in the sense of Harary [2] and Gary Chartrand [1]. Two Graphs G_1 and G_2 are isomorphic if there exists a one-to-one correspondence f from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. By a coloring of a graph, we mean an assignment of colors to the vertices of G such that adjacent vertices are colored differently. The smallest number of colors in any coloring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$. If it is possible to color G from a set of k colors, then G is said to be k-colorable. A coloring that uses k-colors is called a k-coloring.

§2. Ideal Graph of a Graph

In this section, we introduce ideal graph of a graph. We can analyze the properties of graphs by using ideal graph of a graph, which may be of smaller size than the original graph.

Definition 2.1 *For a graph G with sets $\mathscr{C}$ of cycles, $\mathscr{L}$ of longest paths with all the internal vertices of degree* 2*, and $U \subset \mathscr{C}$, $V \subset \mathscr{L}$, its Smarandachely ideal graph $I_d^{U,V}(G)$ of the graph*

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G is formed as follows:

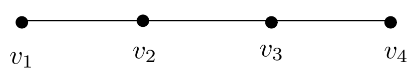
(i) These cycles and the edges lying on a cycle in U *or* $\mathscr{C} \setminus U$ *will remain or not same in Smarandachely ideal graph* $I_d^{U,V}(G)$ *of* G.

(ii) Every longest u-v path in V *or* $\mathscr{L} \setminus V$ *is considered as an edge* uv *or not in Smarandachely ideal graph* $I_d^{U,V}(G)$ *of* G.

Particularly, if $U = \mathscr{C}$ *and* $V = \mathscr{L}$*, i.e., a Smarandachely* $I_d^{\mathscr{C},\mathscr{L}}(G)$ *of* G *is called the ideal graph of* G*,denoted by* $I_d(G)$.

Example 2.2 Some ideal graphs of graphs are shown following.

1.



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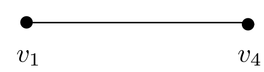
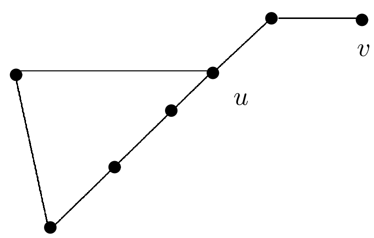


$I_d(G)$

2.



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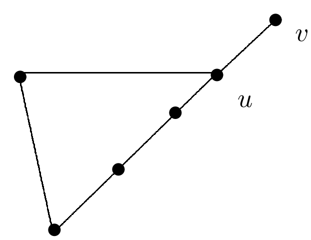


$I_d(G)$

Definition 2.3 *The vertices of the ideal graph* $I_d(G)$ *are called strong vertices of the graph* G *and the vertices, which are not in the ideal graph* $I_d(G)$ *are called weak vertices of the graph* G.

Definition 2.4 *The vanishing number of an edge* uv *of the ideal graph of a graph* G *is defined as the number of internal vertices of the u-v path in the graph* G.

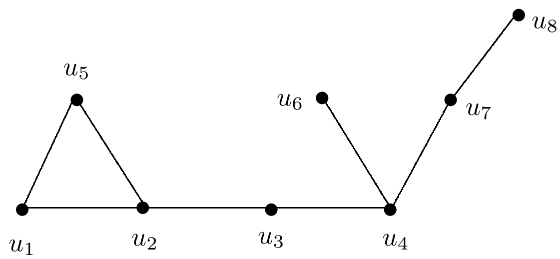
We denote the vanishing number of an edge e of an ideal graph by $v_0(e)$.

Remark 2.5 It is possible to get the original graph G from its ideal graph $I_d(G)$ if we know the vanishing numbers of all the edges of $I_d(G)$.

Definition 2.6 *The vanishing number of the ideal graph of a graph* G *is denoted by* v_{id} *and is defined as the sum of all vanishing numbers of the edges of* $I_d(G)$ *or the number of weak vertices of the graph* G.

Definition 2.7 *The ideal number of a graph* G *is defined as the number of vertices in the ideal graph of the graph* G *or the number of strong vertices of the graph. It is denoted by* p_{id}.

Example 2.8 A graph with its ideal graph is shown in the following. In this graph, the ideal number of the graph G is 6. (i.e. $p_{id} = 6$). Also, in the ideal graph, the vanishing number of the edges are $v_0(u_1u_2) = v_0(u_2u_5) = v_0(u_1u_5) = v_0(u_4u_6) = 0$ and $v_0(u_2u_4) = v_0(u_4u_8) = 1$. The vanishing number($v_{id}$) of the ideal graph $I_d(G)$ is 2.



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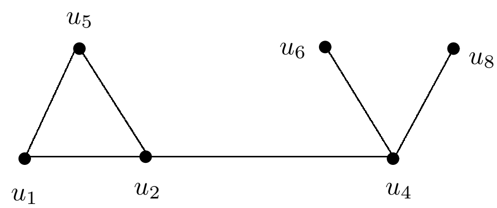


$I_d(G)$

The following proposition is obvious from the above definitions.

Proposition 2.9 *Let G be a graph and $p = |V(G)|$. The following properties are true.*

(i) $p = p_{id} + v_{id}$.

(ii) $p \geq p_{id}$.

(iii) $p = p_{id}$ *if and only if* $G = I_d(G)$.

Proof Proof follows from the Definitions 2.1, 2.6 and 2.7. □

Proposition 2.10 *There are ideal graphs following.*

(i) $I_d(P_n) = P_2$ for every $n \geq 2$.

(ii) $I_d(C_n) = C_n$, $I_d(W_n) = W_n$ and $I_d(K_n) = K_n$ for all n.

(iii) $I_d(K_{1,2}) = P_2$.

(iv) $I_d(K_{m,n}) = K_{m,n}$ except for $K_{1,2}$.

(v) $I_d(G) = G$ if $\delta \geq 3$.

(vi) $I_d(G) = G$ if G is Eulerian.

(vii) $I_d(I_d(G)) = I_d(G)$ for any graph G.

Proof Proof follows from the definition of $I_d(G)$. □

Proposition 2.11 *A vertex v of a graph G is a strong vertex if and only if $deg(v) \leq 1$ or $deg(v) \geq 3$ or the vertex v lies in a cycle.*

Proof Proof follows from the definition of $I_d(G)$. □

Proposition 2.12 *If a vertex v of a graph G is a weak vertex, then $deg(v) = 2$.*

proof Proof follows from the definition of $I_d(G)$. □

Remark 2.13 Converse of the above proposition is not true. For, consider $G = C_3$. Then all the vertices of G are of degree 2 but they are not weak vertices.

§3. Characterization of Connectedness

In this section, we characterize connected graphs using ideal graph.

Theorem 3.1 *A graph G is connected if and only if $I_d(G)$ is connected.*

Proof It is obvious from the definition of $I_d(G)$ that if G is connected, then $I_d(G)$ is connected. Assume that $I_d(G)$ is connected. Let u and v be two vertices of G.

Case i. u and v are strong vertices of G.

Since $I_d(G)$ is connected, there exists an u-v path in $I_d(G)$ that gives an u-v path in G.

Case ii. u is a strong vertex and v is a weak vertex of G.

Then v is an internal vertex of an u_1-v_1 path of G where u_1v_1 is an edge of $I_d(G)$. By assumption there exists an u-u_1 path in $I_d(G)$. Then the paths u-u_1 and u_1-v jointly gives the path in G between u and v.

Case iii. Both u and v are weak vertices of G.

Then u and v are internal vertices of some u_1-w_1 path and u_2-w_2 path in G respectively such that u_1w_1 and u_2w_2 are edges of $I_d(G)$. Then there exists an w_1-u_2 path in $I_d(G)$. Then the paths uw_1u_2v is the required u-v path in G. □

Theorem 3.2 *A graph G and $I_d(G)$ have same number of connected components.*

Proof Proof is obvious from the definition of $I_d(G)$ and Theorem 3.1.

§4. Characterization of Isomorphism

In this section, we characterize isomorphism of two graphs via ideal graphs. Since trees are connected graphs with no cycles, this characterization maybe more useful to analyze the isomorphism of trees.

Lemma 4.1([1]) *If a graph G is isomorphic to a graph G' under a function f, then*

(i) G and G' have same degree sequence

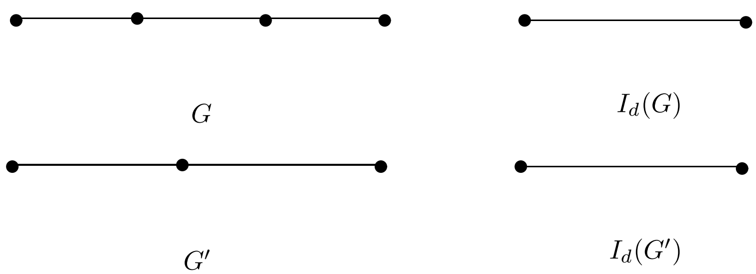
(ii) if G contains a k-cycle for some integer $k \geq 3$, so does G' and

(iii) if G contains a u-v path of length k, then G' contains a $f(u) - f(v)$ path of length k.

Theorem 4.2 *If a graph G is isomorphic to a graph G', then $I_d(G)$ is isomorphic to $I_d(G')$.*

Proof Proof follows from Lemma 4.1. □

Remark 4.3 The following example shows that the converse of the above theorem is not true.



Here, $I_d(G)$ and $I_d(G')$ are isomorphic. But G and G' are not isomorphic.

The following theorem gives the necessary and sufficient condition for two graphs to be isomorphic.

Theorem 4.4 *A graph G is isomorphic to the graph G' if and only if $I_d(G)$ is isomorphic to $I_d(G')$ and the isomorphic edges have same vanishing number.*

Proof Assume the graph G is isomorphic to the graph G'. By Theorem 4.2 and Lemma 4.1, $I_d(G)$ is isomorphic to $I_d(G')$ and the isomorphic edges have same vanishing number. Conversely, assume Id(G) is isomorphic to Id(G') and the isomorphic edges have same vanishing number. If uv and $u'v'$ are isomorphic edges of $I_d(G)$ and $I_d(G')$ respectively with same vanishing number, then the edges uv and $u'v'$ or the paths u-v and u'-v' are isomorphic in G, since they have same vanishing number. Hence G is isomorphic to the graph G'. □

§5. Characterization of Coloring Property

In this section, we give one characterization for 2-colorable and study about the relation between the coloring of ideal graph and the actual graph. Also, we find an upper bound for the chromatic number of a graph.

Theorem 5.1 *A graph G is 2-colorable if and only if $I_d(G)$ is 2-colorable.*

Proof It is obvious from the definition of ideal graph that a graph G has odd cycles if and only if the ideal graph $I_d(G)$ has odd cycles. We know that a graph G is 2-colorable if and only if it contains no odd cycles. Hence a graph G is 2-colorable if and only if $I_d(G)$ is 2-colorable.□

Theorem 5.2 *The strong vertices of a graph G can have the same colors in G and $I_d(G)$ under some 2-coloring if and only if all the edges of $I_d(G)$ have even vanishing number.*

Proof Assume that the strong vertices of a graph G have same colors in G and $I_d(G)$ under some 2-colorings. Let uv be an edge of $I_d(G)$. Then u and v are in different colors in $I_d(G)$ under a 2-coloring. If the vanishing number of uv is an odd number, then u and v have the same colors in G. Thus u or v differs by color in G from $I_d(G)$. This contradicts our assumption. Hence all edges of $I_d(G)$ have even vanishing number. Other part of this theorem is obvious.□

Theorem 5.3 *A graph G is k-colorable with $k \geq 3$ and the strong vertices of G can have the same colors as in $I_d(G)$ under a k-coloring if $I_d(G)$ is k-colorable.*

Proof Let $I_d(G)$ is k-colorable with $k \geq 3$. Assign the same colors for the strong vertices of G as in $I_d(G)$ under a k-coloring. Then for the weak vertices which are lying in the path of connecting strong vertices, we can use 3 colors such that G is k-colorable and the strong vertices of G can have the same colors as in $I_d(G)$. □

Corollary 5.4 *For any graph $G, \chi(G) \leq \chi(I_d(G)) \leq p_{id}$.*

Proof Proof follows from Theorem 5.3. □

References

[1] Gary Chartrand and Ping Zhang, *Introduction to Graph Theory*, Tata McGraw-Hill Publishing Company Limited, 2006.

[2] F. Harary, *Graph Theory*, Addison Wesley, Reading Mass., 1972.