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Smarandache Idempotents in Loop Rings $Z_tL_n(m)$ of the Loops $L_n(m)$

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Abstract In this paper we establish the existence of S-idempotents in case of loop rings $Z_tL_n(m)$ for a special class of loops $L_n(m)$; over the ring of modulo integers Z_t for a specific value of t. These loops satisfy the conditions g_i^2 for every $g_i \in L_n(m)$. We prove $Z_tL_n(m)$ has an S-idempotent when t is a perfect number or when t is of the form 2^ip or 3^ip (where p is an odd prime) or in general when $t = p_1^ip_2$(p_1 and p_2 are distinct odd primes), It is important to note that we are able to prove only the existence of a single S-idempotent; however we leave it as an open problem whether such loop rings have more than one S-idempotent.

§1. Basic Results

This paper has three sections. In section one, we give the basic notions about the loops $L_n(m)$ and recall the definition of S-idempotents in rings. In section two, we establish the existence of S-idempotents in the loop ring $Z_tL_n(m)$. In the final section, we suggest some interesting problems based on our study.

Here we just give the basic notions about the loops $L_n(m)$ and the definition of S-idempotents in rings.

Definition 1.1 [4]. Let R be a ring. An element $x \in R \setminus \{0\}$ is said to be a Smarandache idempotents (S-idempotent) of R if $x^2 = x$ and there exist $a \in R \setminus \{x, 0\}$ such that

$$\begin{aligned} &i. \quad a^2 = x \\ &ii. \quad xa = x \;\text{ or }\; ax = a. \end{aligned}$$

For more about S-idempotent please refer [4].

Definition 1.2 [2]. A positive integer n is said to be a perfect number if n is equal to the sum of all its positive divisors, excluding n itself. e.g. 6 is a perfect number. As $6 = 1 + 2 + 3$.

Definition 1.3 [1]. A non-empty set L is said to form a loop, if in L is defined a binary operation, called product and denoted by $'.'$ such that

1. For $a, b \in L$ we have $a.b \in L$. (closure property.)

2. There exists an element $e \in L$ such that $a.e = e.a = a$ for all $a \in L$. (e is called the identity element of L.)

3. For every ordered pair $(a, b) \in L \times L$ there exists a unique pair $(x, y) \in L \times L$ such that $ax = b$ and $ya = b$.

Definition 1.4 [3]. Let $L_n(m) = \{e, 1, 2, 3, \cdots, n\}$ be a set where $n > 3$, n is odd and m is a positive integer such that $(m, n) = 1$ and $(m - 1, n) = 1$ with $m < n$. Define on $L_n(m)$, a binary operation $'.'$ as following:

$$\begin{array}{ll} i. & e.i = i.e = i \quad \text{for all} \quad i \in L_n(m) \setminus \{e\} \\ ii. & i^2. = e \quad \text{for all} \quad i \in L_n(m) \\ iii. & i.j = t, \quad \text{where} \quad t \equiv (mj - (m-1)i)(\text{mod} n) \quad \text{for all} \quad i, j \in L_n(m), \\ & \quad i \neq e \quad \text{and} \quad j \neq e. \end{array}$$

Then $L_n(m)$ is a loop. This loop is always of even order; further for varying m, we get a class of loops of order $n + 1$ which we denote by

$$L_n = \{L_n(m) | n > 3, n \text{ is odd and } (m, n) = 1, (m - 1, n) = 1 \text{ with } m < n\}.$$

Example 1.1 [3]. Consider $L_5(2) = \{e, 1, 2, 3, 4, 5\}$. The composition table for $L_5(2)$ is given below:

·	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	3	5	2	4
2	2	5	e	4	1	3
3	3	4	1	e	5	2
4	4	3	5	2	e	1
5	5	2	4	1	3	e

This loop is non-commutative and non-associative and of order 6.

§2. Existence of S-idempotents in the Loop Rings $Z_tL_n(m)$

In this section we will prove the existence of an S-idempotent in the loop ring $Z_tL_n(m)$ when t is an even perfect number. Also we will prove that the loop ring $Z_tL_n(m)$ has an S-idempotent when t is of the form 2^ip or 3^ip (where p is an odd prime) or in general when $t = p_1^ip_2$ (p_1 and p_2 are distinct odd primes).

Theorem 2.1. Let $Z_tL_n(m)$ be a loop ring, where t is an even perfect number of the form $t = 2^s(2^{s+1} - 1)$ for some $s > 1$, then $\alpha = 2^s + 2^sg_i \in Z_tL_n(m)$ is an S-idempotent.

Proof. As t is an even perfect number, t must be of the form

$$t = 2^s(2^{s+1} - 1), \qquad \text{for some } s > 1$$

where $2^{s+1} - 1$ is a prime.

Consider

$$\alpha = 2^s + 2^sg_i \in Z_tL_n(m).$$

Choose

$$\beta = (t - 2^s) + (t - 2^s)g_i \in Z_tL_n(m).$$

Clearly

$$\begin{aligned}\alpha^2 &= (2^s + 2^s g_i)^2 \\ &= 2.2^{2s}(1+g_i) \\ &\equiv 2^s(1+g_i) \qquad [2^s.2^{s+1} \equiv 2^s (\text{mod } t)] \\ &= \alpha.\end{aligned}$$

Now

$$\begin{aligned}\beta^2 &= ((t-2^s) + (t-2^s)g_i)^2 \\ &= 2.(t-2^s)^2(1+g_i) \\ &\equiv 2^s(1+g_i) \\ &= \alpha.\end{aligned}$$

Also

$$\begin{aligned}\alpha\beta &= [2^s + 2^s g_i][(t-2^s) + (t-2^s)g_i] \\ &= 2^s(1+g_i)(t-2^s)(1+g_i) \\ &\equiv -2.2^s.2^s(1+g_i) \\ &\equiv (t-2^s)(1+g_i) \\ &= \beta.\end{aligned}$$

So we get

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \qquad \text{and} \quad \alpha\beta = \beta.$$

Therefore $\alpha = 2^s + 2^s g_i$ is an S-idempotent.

Example 2.1. Take the loop ring $Z_6 L_n(m)$. Here 6 is an even perfect number. As $6 = 2.(2^s - 1)$, so $\alpha = 2 + 2g_i$ is an S-idempotent. For

$$\begin{aligned}\alpha^2 &= (2+2g_i)^2 \\ &\equiv 2 + 2g_i \\ &= \alpha.\end{aligned}$$

Choose now

$$\beta = (6-2) + (6-2)g_i.$$

then

$$\begin{aligned}\beta^2 &= (4+4g_i)^2 \\ &\equiv (2+2g_i) \\ &= \alpha.\end{aligned}$$

And

$$\begin{aligned}\alpha\beta &= (2+2g_i)(4+4g_i) \\ &= 8 + 8g_i + 8g_i + 8 \\ &\equiv 4 + 4g_i \\ &= \beta.\end{aligned}$$

So $\alpha = 2 + 2g_i$ is an S-idempotent.

Theorem 2.2. Let $Z_{2p}L_n(m)$ be a loop ring where p is an odd prime such that $p \mid 2^{t_0+1}-1$ for some $t_0 \geq 1$, then $\alpha = 2^{t_0} + 2^{t_0}g_i \in Z_{2p}L_n(m)$ is an S-idempotent.

Proof. Suppose $p \mid 2^{t_0+1} - 1$ for some $t_0 \geq 1$. Take $\alpha = 2^{t_0} + 2^{t_0}g_i \in Z_{2p}L_n(m)$ and $\beta = (2p - 2^{t_0}) + (2p - 2^{t_0})g_i \in Z_{2p}L_n(m)$.

Clearly

$$\begin{aligned} \alpha^2 &= (2^{t_0} + 2^{t_0}g_i)^2 \\ &= 2.2^{2t_0}(1 + g_i) \\ &= 2^{t_0+1}.2^{t_0}(1 + g_i) \\ &\equiv 2^{t_0}(1 + g_i) \\ &= \alpha. \end{aligned}$$

As

$$2^{t_0}.2^{t_0+1} \equiv 2^{t_0}(mod\ 2p)$$

Since

$$2^{t_0+1} \equiv 1(mod\ p)$$

$$\Leftrightarrow 2^{t_0}.2^{t_0+1} \equiv 2^{t_0}(mod\ 2p) \quad \text{for} \quad gcd(2^{t_0}, 2p) = 2, \quad t_0 \geq 1.$$

Also

$$\begin{aligned} \beta^2 &= [(2p - 2^{t_0}) + (2p - 2^{t_0})g_i]^2 \\ &= 2(2p - 2^{t_0})^2(1 + g_i) \\ &\equiv 2.2^{2t_0}(1 + g_i) \\ &= 2^{t_0+1}.2^{t_0}(1 + g_i) \\ &\equiv 2^{t_0}(1 + g_i) \\ &= \alpha. \end{aligned}$$

And

$$\begin{aligned} \alpha\beta &= [2^{t_0} + 2^{t_0}g_i][(2p - 2^{t_0}) + (2p - 2^{t_0})g_i] \\ &\equiv -2^{t_0}(1 + g_i)2^{t_0}(1 + g_i) \\ &= -2.2^{t_0}(1 + g_i) \\ &\equiv (2p - 2^{t_0})(1 + g_i) \\ &= \beta. \end{aligned}$$

So we get

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \qquad \text{and} \quad \alpha\beta = \beta.$$

Hence $\alpha = 2^{t_0} + 2^{t_0}g_i$ is an S-idempotent.

Example 2.2. Consider the loop ring $Z_{10}L_n(m)$. Here $5 \mid 2^{3+1} - 1$, so $t_0 = 3$.

Take

$$\alpha = 2^3 + 2^3 g_i \text{ and } \beta = 2 + 2g_i.$$

Now

$$\begin{aligned} \alpha^2 &= (8+8g_i)^2 \\ &= 64 + 128g_i + 64 \\ &\equiv 8 + 8g_i \\ &= \alpha. \end{aligned}$$

And

$$\begin{aligned} \beta^2 &= (2+2g_i)^2 \\ &= 4 + 8g_i + 4 \\ &\equiv 8 + 8g_i \\ &= \alpha. \end{aligned}$$

Also

$$\begin{aligned} \alpha\beta &= (8+8g_i)(2+2g_i) \\ &= 16 + 16g_i + 16g_i + 16 \\ &\equiv 2 + 2g_i \\ &= \beta. \end{aligned}$$

So $\alpha = 8 + 8g_i$ is an S-idempotent.

Theorem 2.3. Let $Z_{2^i p}L_n(m)$ be a loop ring where p is an odd prime such that $p \mid 2^{t_0+1}-1$ for some $t_0 \geq i$, then $\alpha = 2^{t_0} + 2^{t_0} g_i \in Z_{2^i p}L_n(m)$ is an S-idempotent.

Proof. Note that $p \mid 2^{t_0+1} - 1$ for some $t_0 \geq i$.

Therefore

$$2^{t_0+1} \equiv 1 (mod\ p) \text{ for some } t_0 \geq i$$

$$\Leftrightarrow 2^{t_0}.2^{t_0+1} \equiv 2^{t_0} (mod\ 2^i p) \text{ as } gcd(2^{t_0}, 2^i p) = 2^i, \quad t_0 \geq 1.$$

Now take

$$\alpha = 2^{t_0} + 2^{t_0} g_i \in Z_{2^i p}L_n(m) \text{ and } \beta = (2^i p - 2^{t_0}) + (2^i p - 2^{t_0}) g_i \in Z_{2^i p}L_n(m).$$

Then it is easy to see that

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha \quad \text{and} \quad \alpha\beta = \beta.$$

Hence $\alpha = 2^{t_0} + 2^{t_0} g_i$ is an S-idempotent.

Example 2.3. Take the loop ring $Z_{2^3.7}L_n(m)$. Here $7 \mid 2^{5+1} - 1$, so $t_0 = 5$.

Take

$$\alpha = 2^5 + 2^5 g_i \text{ and } \beta = (2^3.7 - 2^5) + (2^3.7 - 2^5) g_i.$$

Now

$$\begin{aligned}\alpha^2 &= (32+32g_i)^2\\ &= 1024+2048g_i+1024\\ &\equiv 32+32g_i\\ &= \alpha.\end{aligned}$$

And

$$\begin{aligned}\beta^2 &= (24+24g_i)^2\\ &= 576+1152g_i+576\\ &\equiv 24+24g_i\\ &= \alpha.\end{aligned}$$

Also

$$\begin{aligned}\alpha\beta &= (32+32g_i)(24+24g_i)\\ &\equiv 24+24g_i\\ &= \beta.\end{aligned}$$

So $\alpha = 32+32g_i$ is an S-idempotent.

Theorem 2.4. Let $Z_{3^ip}L_n(m)$ be a loop ring where p is an odd prime such that $p \mid 2.3^{t_0}-1$ for some $t_0 \geq i$, then $\alpha = 3^{t_0}+3^{t_0}g_i \in Z_{3^ip}L_n(m)$ is an S-idempotent.

Proof. Suppose $p \mid 2.3^{t_0}-1$ for some $t_0 \geq i$.

Take

$$\alpha = 3^{t_0}+3^{t_0}g_i \in Z_{3^ip}L_n(m) \text{ and } \beta = (3^ip-3^{t_0})+(3^ip-3^{t_0})g_i \in Z_{3^ip}L_n(m).$$

Then

$$\begin{aligned}\alpha^2 &= (3^{t_0}+3^{t_0}g_i)^2\\ &= 2.3^{2t_0}(1+g_i)\\ &= 2.3^{t_0}3^{t_0}(1+g_i)\\ &\equiv 3^{t_0}(1+g_i)\\ &= \alpha.\end{aligned}$$

As

$$2.3^{t_0} \equiv 1(mod\ p) \text{ for some } t_0 \geq i$$

$$\Leftrightarrow 2.3^{t_0}.3^{t_0} \equiv 3^{t_0}(mod\ 3^ip) \text{ as } gcd(3^{t_0},3^ip) = 3^i, \quad t_0 \geq 1.$$

Similarly

$$\beta^2 = \alpha \quad \text{and} \quad \alpha\beta = \beta.$$

So $\alpha = 3^{t_0}+3^{t_0}g_i$ is an S-idempotent.

Example 2.4. Take the loop ring $Z_{3^2.5}L_n(m)$. Here $5 \mid 2.3^5 - 1$, so $t_0 = 5$.

Take

$$\alpha = 3^5 + 3^5 g_i \ \text{ and } \ \beta = (3^2.5 - 3^5) + (3^2.5 - 3^5)g_i.$$

Now

$$\begin{aligned} \alpha^2 &= (18 + 18g_i)^2 \\ &\equiv 18 + 18g_i \\ &= \alpha. \end{aligned}$$

And

$$\begin{aligned} \beta^2 &= (27 + 27g_i)^2 \\ &\equiv 18 + 18g_i \\ &= \alpha. \end{aligned}$$

Also

$$\alpha\beta = \beta.$$

So $\alpha = 3^5 + 3^5 g_i$ is an S-idempotent.

We can generalize Theorem 2.3 and Theorem 2.4 as following:

Theorem 2.5. Let $Z_{p_1^i p_2}L_n(m)$ be a loop ring where p_1 and p_2 are distinct odd primes and $p_2 \mid 2.p_1^{t_0} - 1$ for some $t_0 \geq i$, then $\alpha = p_1^{t_0} + p_1^{t_0} g_i \in Z_{p_1^i p_2}L_n(m)$ is an S-idempotent.

Proof. Suppose $p_2 \mid 2.p_1^{t_0} - 1$ for some $t_0 \geq i$.

Take

$$\alpha = p_1^{t_0} + p_1^{t_0} g_i \in Z_{p_1^i p_2}L_n(m) \ \text{ and } \ \beta = (p_1^i p_2 - p_1^{t_0}) + (p_1^i p_2 - p_1^{t_0})g_i \in Z_{p_1^i p_2}L_n(m).$$

Then

$$\begin{aligned} \alpha^2 &= (p_1^{t_0} + p_1^{t_0} g_i)^2 \\ &= 2.p_1^{2t_0}(1 + g_i) \\ &= 2.p_1^{t_0} p_1^{t_0}(1 + g_i) \\ &\equiv p_1^{t_0}(1 + g_i) \\ &= \alpha. \end{aligned}$$

As

$$2.p_1^{t_0} \equiv 1 (mod\ p_2) \ \text{ for some } \ t_0 \geq i$$

.

$$\Leftrightarrow 2.p_1^{t_0}.p_1^{t_0} \equiv p_1^{t_0}(mod\ p_1^i p_2) \ \text{ as } \ gcd(p_1^{t_0}, p_1^i p_2) = p_1^i, \ \ t_0 \geq i.$$

Similarly

$$\beta^2 = \alpha \quad \text{ and } \ \alpha\beta = \beta.$$

So $\alpha = p_1^{t_0} + p_1^{t_0} g_i$ is an S-idempotent.

§3. Conclusion

We see in all the 5 cases described in the Theorem 2.1 to 2.5 we are able to establish the existence of one non-trivial S-idempotent. however we are not able to prove the uniqueness of this S-idempotent. Hence we suggest the following problems:

• Does the loop rings described in the Theorems 2.1 to 2.5 can have more than one S-idempotent?

• Does the loop rings $Z_tL_n(m)$ have S-idempotent when t is of the form $t = p_1p_2 \dots p_s$ where $p_1p_2 \dots p_s$ are distinct odd primes?

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