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Some identities involving the near pseudo Smarandache function

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Abstract For any positive integer n and fixed integer $t \geq 1$, we define function $U_t(n) = \min\{k : 1^t + 2^t + \cdots + n^t + k = m,\ n \mid m,\ k \in N^+,\ t \in N^+\}$, where $n \in N^+$, $m \in N^+$, which is a new pseudo Smarandache function. The main purpose of this paper is using the elementary method to study the properties of $U_t(n)$, and obtain some interesting identities involving function $U_t(n)$.

Keywords Some identities, reciprocal, pseudo Smarandache function.

§1. Introduction and results

In reference [1], A.W.Vyawahare defined the near pseudo Smarandache function $K(n)$ as $K(n) = m = \frac{n(n+1)}{2} + k$, where k is the small positive integer such that n divides m. Then he studied the elementary properties of $K(n)$, and obtained a series interesting results for $K(n)$. For example, he proved that $K(n) = \frac{n(n+3)}{2}$, if n is odd, and $K(n) = \frac{n(n+2)}{2}$, if n is even; The equation $K(n) = n$ has no positive integer solution. In reference [2], Zhang Yongfeng studied the calculating problem of an infinite series involving the near pseudo Smarandache function $K(n)$, and proved that for any real number $s > \frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{1}{K^s(n)}$ is convergent, and

$$\sum_{n=1}^{\infty} \frac{1}{K(n)} = \frac{2}{3} \ln 2 + \frac{5}{6},$$

$$\sum_{n=1}^{\infty} \frac{1}{K^2(n)} = \frac{11}{108}\pi^2 - \frac{22 + 2\ln 2}{27}.$$

Yang hai and Fu Ruiqin [3] studied the mean value properties of the near pseudo Smarandache function $K(n)$, and obtained two asymptotic formula by using the analytic method. They proved that for any real number $x \geq 1$,

$$\sum_{n \leq x} d(k) = \sum_{n \leq x} d\left(K(n) - \frac{n(n+1)}{2}\right) = \frac{3}{4} x \log x + Ax + O\left(x^{\frac{1}{2}} \log^2 x\right),$$

where A is a computable constant.

$$\sum_{n\leq x}\varphi\left(K(n)-\frac{n(n+1)}{2}\right)=\frac{93}{28\pi^2}x^2+O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number.

In this paper, we define a new near Smarandache function $U_t(n)=\min\{k:\ 1^t+2^t+\cdots+n^t+k=m,\ n\mid m,\ k\in N^+,\ t\in N^+\}$, where $n\in N^+$, $m\in N^+$. Then we study its elementary properties. About this function, it seems that none had studied it yet, at least we have not seen such a paper before. In this paper, we using the elementary method to study the calculating problem of the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{U_t^s(n)},$$

and give some interesting identities. That is, we shall prove the following:

Theorem 1. For any real number $s>1$, we have the identity

$$\sum_{n=1}^{\infty}\frac{1}{U_1^s(n)}=\zeta(s)\left(2-\frac{1}{2^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Theorem 2. For any real number $s>1$, we have

$$\sum_{n=1}^{\infty}\frac{1}{U_2^s(n)}=\zeta(s)\left[1+\frac{1}{5^s}-\frac{1}{6^s}+2\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\right].$$

Theorem 3. For any real number $s>1$, we also have

$$\sum_{n=1}^{\infty}\frac{1}{U_3^s(n)}=\zeta(s)\left[1+\left(1-\frac{1}{2^s}\right)^2\right].$$

Taking $s=2,\ 4$, and note that $\zeta(2)=\frac{\pi^2}{6}$, $\zeta(4)=\frac{\pi^4}{90}$, from our theorems we may immediately deduce the following:

Corollary. Let $U_t(n)$ defined as the above, then we have the identities

$$\sum_{n=1}^{\infty}\frac{1}{U_1^2(n)}=\frac{7}{24}\pi^2;\qquad \sum_{n=1}^{\infty}\frac{1}{U_2^2(n)}=\frac{2111}{5400}\pi^2;$$

$$\sum_{n=1}^{\infty}\frac{1}{U_3^2(n)}=\frac{25}{96}\pi^2;\qquad \sum_{n=1}^{\infty}\frac{1}{U_1^4(n)}=\frac{31}{1440}\pi^4;$$

t

$$\sum_{n=1}^{\infty}\frac{1}{U_2^4(n)}=\frac{2310671}{72900000}\pi^4;\qquad \sum_{n=1}^{\infty}\frac{1}{U_3^4(n)}=\frac{481}{23040}\pi^4.$$

§2. Some lemmas

To complete the proof of the theorems, we need the following several lemmas.

Lemma 1. For any positive integer n, we have

$$U_1(n) = \begin{cases} \frac{n}{2}, & \text{if } 2 \mid n, \\ n, & \text{if } 2 \dagger n. \end{cases}$$

Proof. See reference [1].

Lemma 2. For any positive integer n, we also have

$$U_2(n) = \begin{cases} \frac{5}{6}n, & \text{if } \quad n \equiv 0(\text{mod } 6), \\ n, & \text{if } \quad n \equiv 1(\text{mod } 6) \quad \text{or} \quad n \equiv 5(\text{mod } 6), \\ \frac{n}{2}, & \text{if } \quad n \equiv 2(\text{mod } 6) \quad \text{or} \quad n \equiv 4(\text{mod } 6), \\ \frac{n}{3}, & \text{if } \quad n \equiv 3(\text{mod } 6). \end{cases}$$

Proof. It is clear that

$$\begin{aligned} U_2(n) &= \min\{k : 1^2 + 2^2 + \cdots + n^2 + k = m, n \mid m, k \in N^+\} \\ &= \min\{k : \frac{n(n+1)(2n+1)}{6} + k \equiv 0(\text{mod } n), k \in N^+\}. \end{aligned}$$

(1) If $n \equiv 0(\text{mod } 6)$, then we have $n = 6h_1(h_1 = 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{6h_1(6h_1+1)(12h_1+1)}{6} \\ &= 72h_1^3 + 18h_1^2 + h_1, \end{aligned}$$

so $n \mid \frac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $6h_1 \mid h_1 + U_2(n)$, then $U_2(n) = \frac{5n}{6}$.

(2) If $n \equiv 1(\text{mod } 6)$, then we have $n = 6h_2 + 1(h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+1)(6h_2+2)(12h_2+3)}{6} \\ &= 12h_2^2(6h_2+1) + 7h_2(6h_2+1) + 6h_2 + 1, \end{aligned}$$

because $n \mid \frac{n(n+1)(2n+1)}{6}$, so $n \mid \frac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $n \mid U_2(n)$, then $U_2(n) = n$.

If $n \equiv 5(\text{mod } 6)$, then we have $n = 6h_2 + 5(h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+5)(6h_2+6)(12h_2+11)}{6} \\ &= 12h_2^2(6h_2+5) + 23h_2(6h_2+5) + 11(6h_2+5), \end{aligned}$$

because $n \mid \frac{n(n+1)(2n+1)}{6}$, so $n \mid \frac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $n \mid U_2(n)$, then $U_2(n) = n$.

(3) If $n \equiv 2(\text{mod } 6)$, then we have $n = 6h_2 + 2(h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+2)(6h_2+3)(12h_2+5)}{6} \\ &= 12h_2^2(6h_2+2) + 11h_2(6h_2+2) + 2(6h_2+2) + 3h_2 + 1, \end{aligned}$$

so $n \mid \dfrac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $6h_2 + 2 \mid 3h_2 + 1 + U_2(n)$, then $U_2(n) = \dfrac{n}{2}$.

If $n \equiv 4 (\bmod\ 6)$, then we have $n = 6h_2 + 4 (\ h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+4)(6h_2+5)(12h_2+9)}{6} \\ &= 12h_2^2(6h_2+4) + 19h_2(6h_2+4) + 7(6h_2+4) + 3h_2 + 3, \end{aligned}$$

so $n \mid \dfrac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $2(3h_2+2) \mid 3h_2 + 2 + U_2(n)$, then $U_2(n) = \dfrac{n}{2}$.

(4) If $n \equiv 3 (\bmod\ 6)$, then we have $n = 6h_2 + 3 (\ h_2 = 0, 1, 2 \cdots)$,

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{(6h_2+3)(6h_2+4)(12h_2+7)}{6} \\ &= 12h_2^2(6h_2+3) + 15h_2(6h_2+3) + 4(6h_2+3) + 4h_2 + 2, \end{aligned}$$

so $n \mid \dfrac{n(n+1)(2n+1)}{6} + U_2(n)$ if and only if $3(2h_2+1) \mid 2(2h_2+2) + U_2(n)$, then $U_2(n) = \dfrac{n}{3}$.

Combining (1), (2), (3) and (4) we may immediately deduce Lemma 2.

Lemma 3. For any positive integer n, we have

$$U_3(n) = \begin{cases} \dfrac{n}{2}, & \text{if} \quad n \equiv 2 (\bmod\ 4), \\ n, & \text{otherwise}. \end{cases}$$

Proof. From the definition of $U_3(n)$ we have

$$\begin{aligned} U_3(n) &= \min\{k : 1^3 + 2^3 + \cdots + n^3 + k = m, n \mid m, k \in N^+\} \\ &= \min\{k : \frac{n^2(n+1)^2}{4} + k \equiv 0 (\bmod\ n), k \in N^+\}. \end{aligned}$$

(a) If $n \equiv 2 (\bmod\ 4)$, then we have $n = 4h_1 + 2 (\ h_1 = 0, 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = (4h_1+2)^3(2h_1+1) + (4h_1+2)^2(2h_1+1) + (2h_1+1)^2,$$

so $n \mid \dfrac{n^2(n+1)^2}{4}$ if and only if $2(2h_1+1) \mid (2h_1+1)^2 + U_3(n)$, then $U_3(n) = \dfrac{n}{2}$.

(b) If $n \equiv 0 (\bmod\ 4)$, then we have $n = 4h_2 (\ h_2 = 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = 4h_2^2(4h_2+1)^2,$$

so $n \mid \dfrac{n^2(n+1)^2}{4} + U_3(n)$ if and only if $n \mid U_3(n)$, then $U_3(n) = n$.

If $n \equiv 1 (\bmod\ 4)$, then we have $n = 4h_1 + 1 (\ h_1 = 0, 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = (4h_1+1)^2(2h_1+1)^2,$$

so $n \mid \dfrac{n^2(n+1)^2}{4} + U_3(n)$ if and only if $n \mid U_3(n)$, then $U_3(n) = n$.

If $n \equiv 3 (\bmod\ 4)$, then we have $n = 4h_1 + 3 (\ h_1 = 0, 1, 2 \cdots)$,

$$\frac{n^2(n+1)^2}{4} = 4(4h_1+3)^2(h_1+1)^2,$$

uso $n \mid \dfrac{n^2(n+1)^2}{4} + U_3(n)$ if and only if $n \mid U_3(n)$, then $U_3(n) = n$.

Now Lemma 3 follows from (a) and (b).

§3. Proof of the theorems

In this section, we shall use the elementary methods to complete the proof of the theorems. First we prove Theorem 1. For any real number $s > 1$, from Lemma 1 we have

$$\sum_{n=1}^{\infty} \frac{1}{U_1^s(n)} = \sum_{\substack{h=1 \\ n=2h}}^{\infty} \frac{1}{(\frac{n}{2})^s} + \sum_{\substack{h=0 \\ n=2h+1}}^{\infty} \frac{1}{n^s} = \sum_{h=1}^{\infty} \frac{1}{h^s} + \sum_{h=0}^{\infty} \frac{1}{(2h+1)^s} = \zeta(s)\left(2 - \frac{1}{2^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function. This proves Theorem 1.

For $t = 2$ and real number $s > 1$, from Lemma 2 we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{U_2^s(n)} &= \sum_{\substack{h_1=1 \\ n=6h_1}}^{\infty} \frac{1}{\left(\frac{5n}{6}\right)^s} + \sum_{\substack{h_2=0 \\ n=6h_2+1}}^{\infty} \frac{1}{n^s} + \sum_{\substack{h_2=0 \\ n=6h_2+2}}^{\infty} \frac{1}{\left(\frac{n}{3}\right)^s} + \sum_{\substack{h_2=0 \\ n=6h_2+4}}^{\infty} \frac{1}{\left(\frac{n}{2}\right)^s} + \sum_{\substack{h_2=0 \\ n=6h_2+5}}^{\infty} \frac{1}{n^s} \\
&= \sum_{h_1=1}^{\infty} \frac{1}{(5h_1)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(6h_2+1)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(3h_2+1)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(2h_2+1)^s} + \\
&\quad \sum_{h_2=0}^{\infty} \frac{1}{(3h_2+2)^s} + \sum_{h_2=0}^{\infty} \frac{1}{(6h_2+5)^s} \\
&= \zeta(s)\left[1 + \frac{1}{5^s} - \frac{1}{6^s} + 2\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\right],
\end{aligned}$$

This completes the proof of Theorem 2.

If $t = 3$, then for any real number $s > 1$, from Lemma 3 we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{U_3^s(n)} &= \sum_{\substack{h_2=1 \\ n=4h_2}}^{\infty} \frac{1}{n^s} + \sum_{\substack{h_1=0 \\ n=4h_1+1}}^{\infty} \frac{1}{n^s} + \sum_{\substack{h_1=0 \\ n=4h_1+2}}^{\infty} \frac{1}{\left(\frac{n}{2}\right)^s} + \sum_{\substack{h_1=0 \\ n=4h_1+3}}^{\infty} \frac{1}{n^s} \\
&= \sum_{h_2=1}^{\infty} \frac{1}{(4h_2)^s} + \sum_{h_1=0}^{\infty} \frac{1}{(4h_1+1)^s} + \sum_{h_1=0}^{\infty} \frac{1}{(2h_1+1)^s} + \sum_{h_1=0}^{\infty} \frac{1}{(4h_1+3)^s} \\
&= \zeta(s)\left[1 + \left(1 - \frac{1}{2^s}\right)^2\right],
\end{aligned}$$

This completes the proof of Theorem 3.

Open Problem. For any integer $t > 3$ and real number $s > 1$, whether there exists a calculating formula for the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{U_t^s(n)} \ ?$$

This is an open problem.

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