Scientia Magna
Vol. 3 (2007), No. 3, 26-29

An identity involving the function $e_p(n)$

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Abstract The main purpose of this paper is to study the relationship between the Riemann zeta-function and an infinite series involving the Smarandache function $e_p(n)$ by using the elementary method, and give an interesting identity.

Keywords Riemann zeta-function, infinite series, identity.

§1. Introduction and Results

Let p be any fixed prime, n be any positive integer, $e_p(n)$ denotes the largest exponent of power p in n. That is, $e_p(n) = m$, if $p^m \mid n$ and $p^{m+1} \nmid n$. In problem 68 of [1], Professor F.Smarandache asked us to study the properties of the sequence $\{e_p(n)\}$. About the elementary properties of this function, many scholars have studied it (see reference [2]-[7]), and got some useful results. For examples, Liu Yanni [2] studied the mean value properties of $e_p(b_k(n))$, where $b_k(n)$ denotes the k-th free part of n, and obtained an interesting mean value formula for it. That is, let p be a prime, k be any fixed positive integer, then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} e_p(b_k(n)) = \left(\frac{p^k - p}{(p^k-p)(p-1)} - \frac{k-1}{p^k-1}\right) x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number.

Wang Xiaoying [3] studied the mean value properties of $\sum\limits_{n\leq x}((n+1)^m - n^m)e_p(n)$, and proved the following conclusion:

Let p be a prime, $m \geq 1$ be any integer, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}((n+1)^m - n^m)e_p(n) = \frac{1}{p-1}\frac{m}{m+1}x + O\left(x^{1-\frac{1}{m}}\right).$$

Gao Nan [4] and [5] also studied the mean value properties of the sequences $p^{e_q(n)}$ and $p^{e_q(b(n))}$, got two interesting asymptotic formulas:

$$\sum_{n\leq x} p^{e_q(n)} = \begin{cases} \frac{q-1}{q-p}x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \text{if } \ q > p; \\ \frac{p-1}{p\ln p}x\ln x + \left(\frac{p-1}{p\ln p}(\gamma - 1) + \frac{p+1}{2p}\right)x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \text{if } \ q = p. \end{cases}$$

and

$$\sum_{n\leq x} p^{e_q(b(n))} = \frac{q^2+p^2q+p}{q^2+q+1}x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where ϵ is any fixed positive number, γ is the Euler constant.

Lv Chuan [6] used elementary and analytic methods to study the asymptotic properties of $\sum_{n\leq x} e_p(n)\varphi(n)$ and obtain an interesting asymptotic formula:

$$\sum_{n\leq x} e_p(n)\varphi(n) = \frac{3p}{(p+1)\pi^2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

Ren Ganglian [7] studied the properties of the sequence $e_p(n)$ and give some sharper asymptotic formulas for the mean value $\sum_{n\leq x} e_p^k(n)$.

Especially in [8], Xu Zhefeng studied the elementary properties of the primitive numbers of power p, and got an useful result. That is, for any prime p and complex number s, we have the identity:

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s-1}.$$

In this paper, we shall use the elementary methods to study the relationship between the Riemann zeta-function and an infinite series involving $e_p(n)$, and obtain an interesting identity. That is, we shall prove the following conclusion:

Theorem. For any prime p and complex number s with $Re(s) > 1$, we have the identity

$$\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s-1},$$

where $\zeta(s)$ is the Riemann zeta-function.

From this theorem, we can see that $\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)}$ denote the same Dirichlet series. Of course, we can also obtain some relationship between $\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s}$ and $\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)}$, that is, we have the following conclusion:

Corollary. For any prime p, we have

$$e_p(m) = \sum_{\substack{n\in N \\ S_P(n)=m}} 1.$$

§2. Proof of the theorem

In this section, we shall use elementary methods to complete the proof of the theorem.

Let $m = e_p(n)$, if $p^m \mid\mid n$, then we can write $n = p^m n_1$, where $(n_1, p) = 1$. Noting that, $e_p(n)$ is the largest exponent of power p, so we have

$$\sum_{n=1}^{\infty} \frac{e_p(n)}{n^s} = \sum_{m=1}^{\infty} \sum_{\substack{n_1=1\\(n_1,p)=1}}^{\infty} \frac{m}{(p^m n_1)^s} = \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \sum_{\substack{n_1=1\\p\nmid n_1}}^{\infty} \frac{1}{n_1^s} = \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\sum_{n_1=1}^{\infty} \frac{1}{n_1^s} - \sum_{\substack{n_1=1\\p\mid n_1}}^{\infty} \frac{1}{n_1^s} \right), \quad (1)$$

let $n_1 = pn_2$, then

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\sum_{n_1=1}^{\infty} \frac{1}{n_1^s} - \sum_{\substack{n_1=1\\p\mid n_1}}^{\infty} \frac{1}{n_1^s} \right) &= \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\zeta(s) - \sum_{n_2=1}^{\infty} \frac{1}{p^s n_2^s} \right) \\ &= \sum_{m=1}^{\infty} \frac{m}{p^{ms}} \left(\zeta(s) - \zeta(s) \frac{1}{p^s} \right) \\ &= \zeta(s) \left(1 - \frac{1}{p^s} \right) \sum_{m=1}^{\infty} \frac{m}{p^{ms}}. \end{aligned}$$

Since

$$\sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \frac{1}{p^s} + \sum_{m=1}^{\infty} \frac{m+1}{p^{(m+1)s}},$$

$$\frac{1}{p^s} \cdot \sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \sum_{m=1}^{\infty} \frac{m}{p^{(m+1)s}},$$

then

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{m}{p^{ms}} - \frac{1}{p^s} \cdot \sum_{m=1}^{\infty} \frac{m}{p^{ms}} &= \frac{1}{p^s} + \sum_{m=1}^{\infty} \frac{m+1}{p^{(m+1)s}} - \sum_{m=1}^{\infty} \frac{m}{p^{(m+1)s}} \\ &= \frac{1}{p^s} + \sum_{m=1}^{\infty} \frac{1}{p^{(m+1)s}} = \sum_{m=1}^{\infty} \frac{1}{p^{ms}}. \end{aligned}$$

That is,

$$\left(1 - \frac{1}{p^s} \right) \sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \sum_{m=1}^{\infty} \frac{1}{p^{ms}} = \frac{1}{p^s} \frac{1}{1 - \frac{1}{p^s}},$$

so

$$\sum_{m=1}^{\infty} \frac{m}{p^{ms}} = \frac{1}{p^s \left(1 - \frac{1}{p^s} \right)^2}. \quad (2)$$

Now, combining (1) and (2), we have the following identity

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{e_p(n)}{n^s} &= \sum_{m=1}^{\infty}\frac{m}{p^{ms}}\left(\sum_{n_1=1}^{\infty}\frac{1}{n_1^s}-\sum_{\substack{n_1=1\\ p\mid n_1}}^{\infty}\frac{1}{n_1^s}\right)\\ &= \zeta(s)\left(1-\frac{1}{p^s}\right)\sum_{m=1}^{\infty}\frac{m}{p^{ms}}\\ &= \zeta(s)\left(1-\frac{1}{p^s}\right)\frac{1}{p^s\left(1-\frac{1}{p^s}\right)^2}=\frac{\zeta(s)}{p^s-1}.\end{aligned}$$

This completes the proof of Theorem.

Then, noting the definition and properties of $S_p(n)$, we have

$$\sum_{n=1}^{\infty}\frac{1}{S_p^s(n)}=\sum_{m=1}^{\infty}\frac{1}{(pm)^s}\sum_{\substack{n\in N\\ S_P(n)=mp}}1, \tag{3}$$

and we also have

$$\sum_{n=1}^{\infty}\frac{e_p(n)}{n^s}=\sum_{m=1}^{\infty}\frac{e_p(mp)}{(mp)^s},$$

therefore, from the definition of $e_p(n)$, we can easily get

$$\sum_{m=1}^{\infty}\frac{e_p(mp)}{(mp)^s}=\sum_{m=1}^{\infty}\frac{1}{(pm)^s}\sum_{\substack{n\in N\\ S_P(n)=mp}}1. \tag{4}$$

Combining (3) and (4), it is clear that

$$e_p(m)=\sum_{\substack{n\in N\\ S_P(n)=m}}1.$$

This completes the proof of Corollary.

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