On an inequality for the Smarandache function

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1. In paper [2] the author proved among others the inequality $S(ab) \leq aS(b)$ for all a, b positive integers. This was refined to

$$S(ab) \leq S(a) + S(b) \tag{1}$$

in [1]. Our aim is to show that certain results from our recent paper [3] can be obtained in a simpler way from a generalization of relation (1). On the other hand, by the method of Le [1] we can deduce similar, more complicated inequalities of type (1).

2. By mathematical induction we have from (1) immediately:

$$S(a_1 a_2 \ldots a_n) \leq S(a_1) + S(a_2) + \ldots + S(a_n) \tag{2}$$

for all integers $a_i \geq 1$ $(i = 1, \ldots, n)$. When $a_1 = \ldots = a_n = n$ we obtain

$$S(a^n) \leq nS(a). \tag{3}$$

For three applications of this inequality, remark that

$$S((m!)^n) \leq nS(m!) = nm \tag{4}$$

since $S(m!) = m$. This is inequality 3) part 1. from [3]. By the same way, $S((n!)^{(n-1)!}) \leq (n-1)!S(n!) = (n-1)!n = n!$, i.e.

$$S((n!)^{(n-1)!}) \leq n! \tag{5}$$

Inequality (5) has been obtained in [3] by other arguments (see 4) part 1.).

Finally, by $S(n^2) \leq 2S(n) \leq n$ for n even (see [3], inequality 1), $n > 4$, we have obtained a refinement of $S(n^2) \leq n$:

$$S(n^2) \leq 2S(n) \leq n \tag{6}$$

for $n > 4$, even.

3. Let m be a divisor of n, i.e. $n = km$. Then (1) gives $S(n) = S(km) \leq S(m) + S(k)$, so we obtain:

If $m|n$, then

$$S(n) - S(m) \leq S\left(\frac{n}{m}\right). \tag{7}$$

As an application of (7), let $d(n)$ be the number of divisors of n. Since $\prod_{k|n} k = n^{d(n)/2}$, and $\prod_{k \leq n} k = n!$ (see [3]), and by $\prod_{k|n} k | \prod_{k \leq n} k$, from (7) we can deduce that

$$S(n^{d(n)/2}) + S(n!/n^{d(n)/2}) \geq n. \tag{8}$$

This improves our relation (10) from [3].

4. Let $S(a) = u$, $S(b) = v$. Then $b|v!$ and $u!|x(x-1)\ldots(x-u+1)$ for all integers $x \geq u$. But from $a|u!$ we have $a|x(x-1)\ldots(x-u+1)$ for all $x \geq u$. Let $x = u+v+k$ $(k \geq 1)$. Then, clearly $ab(v+1)\ldots(v+k)|(u+v+k)!$, so we have $S[ab(v+1)\ldots(v+k)] \leq u+v+k$. Here $v = S(b)$, so we have obtained that

$$S[ab(S(b)+1)\ldots(S(b)+k)] \leq S(a) + S(b) + k. \tag{9}$$

For example, for $k = 1$ one has

$$S[ab(S(b)+1)] \leq S(a) + S(b) + 1. \tag{10}$$

This is not a consequence of (2) for $n = 3$, since $S[S(b)+1]$ may be much larger than 1.

References

[1] M. Le, *An inequality concerning the Smarandache function*, Smarandache Notions J., vol. **9**(1998), 124-125.

[2] J. Sándor, *On certain inequalities involving the Smarandache function*, Smarandache Notions J., vol. **7**(1996), 3-6.

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