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Inequalities for the polygamma functions with application[1]

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Abstract We present some inequalities for the polygamma funtions. As an application, we give the upper and lower bounds for the expression $\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma$, where $\gamma=0.57721\cdots$ is the Euler's constant.

Keywords Inequality; Polygamma function; Harmonic sequence; Euler's constant.

§1. Inequalities for the Polygamma Function

The gamma function is usually defined for $Rez>0$ by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt.$$

The psi or digamma function, the logarithmic derivative of the gamma function and the polygamma functions can be expressed as

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma+\sum_{k=0}^{\infty}\left(\frac{1}{1+k}-\frac{1}{z+k}\right),$$

$$\psi^n(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}}$$

for $Rez>0$ and $n=1,2,\cdots$, where $\gamma=0.57721\cdots$ is the Euler's constant.

M. Merkle [2] established the inequality

$$\frac{1}{x}+\frac{1}{2x^2}+\sum_{k=1}^{2N}\frac{B_{2k}}{x^{2k+1}}<\sum_{k=0}^{\infty}\frac{1}{(x+k)^2}<\frac{1}{x}+\sum_{k=1}^{2N+1}\frac{B_{2k}}{x^{2k+1}}$$

for all real $x>0$ and all integers $N\geq 1$, where B_k denotes Bernoulli numbers, defined by

$$\frac{t}{e^t-1}=\sum_{j=0}^{\infty}\frac{B_j}{j!}t^j.$$

The first five Bernoulli numbers with even indices are

$$B_2=\frac{1}{6},B_4=-\frac{1}{30},B_6=\frac{1}{42},B_8=-\frac{1}{30},B_{10}=\frac{5}{66}.$$

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The following theorem 1 establishes a more general result.

Theorem 1. Let $m \geq 0$ and $n \geq 1$ be integers, then we have for $x > 0$,

$$\ln x - \frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x) < \ln x - \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} \tag{1}$$

and

$$\begin{aligned} &\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}} \\ < (-1)^{n+1}\psi^{(n)}(x) < &\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}. \end{aligned} \tag{2}$$

Proof. From Binet's formula [6, p. 103]

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) \frac{e^{-xt}}{t^2} dt,$$

we conclude that

$$\psi(x) = \ln x - \frac{1}{2x} - \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) \frac{e^{-xt}}{t} dt \tag{3}$$

and therefore

$$(-1)^{n+1}\psi^{(n)}(n) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) t^{n-1} e^{-xt} dt. \tag{4}$$

It follows from Problem 154 in Part I, Chapter 4, of [3] that

$$\sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} t^{2j} < \frac{t}{e^t - 1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} t^{2j} \tag{5}$$

for all integers $m > 0$. The inequality (5) can be also found in [4].

From (3) and (5) we conclude (1), and we obtain (2) from (4) and (5). This completes the proof of the theorem 1.

Note that $\psi(x+1) = \psi(x) + \frac{1}{x}$ (see [1, p. 258]), (1) can be written as

$$\frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x+1) - \ln x < \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} \tag{6}$$

and (2) can be written as

$$\begin{aligned} &\frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}} \\ < (-1)^{n+1}\psi^{(n)}(x) < &\frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}. \end{aligned} \tag{7}$$

In particular, taking in (6) $m = 0$ we obtain for $x > 0$,

$$\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x} \tag{8}$$

and taking in (7) $m = 1$ and $n = 1$, we obtain for $x > 0$

$$\frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} < \frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \tag{9}$$

The inequalities (8) and (9) play an important role in the proof of the theorem 2 in Section 2.

§2. Inequalities for Euler's Constant

Euler's constant $\gamma = 0.57721\cdots$ is defined by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right).$$

It is of interest to investigate the bounds for the expression $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln n - \gamma$. The inequality

$$\frac{1}{2n} - \frac{1}{8n^2} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n}$$

is called in literature Franel's inequality [3, Ex. 18].

It is given in [1, p. 258] that $\psi(n) = \sum\limits_{k=1}^{n-1} \frac{1}{k} - \gamma$, and then we have get

$$\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma = \psi(n+1) - \ln n. \tag{10}$$

Taking in (6) $x = n$ we obtain that

$$\frac{1}{2n} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j}\frac{1}{n^{2j}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j}\frac{1}{n^{2j}}. \tag{11}$$

The inequality (11) provides closer bounds for $\sum\limits_{k=1}^{n} \frac{1}{k} - \ln n - \gamma$.

L.Tóth [5, p. 264] proposed the following problems:

(i) Prove that for every positive integer n we have

$$\frac{1}{2n + \frac{2}{5}} < \sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}.$$

(ii) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ can not be replaced by a slightly larger number.

The following Theorem 2 answers the problem due to L.Tóth.

Theorem 2. For every positive integer n,

$$\frac{1}{2n + a} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + b}, \tag{12}$$

with the best possible constants

$$a = \frac{1}{1-\gamma} - 2 \qquad \text{and} \qquad b = \frac{1}{3}$$

Proof. By (10), the inequality (12) can be rearranged as

$$b < \frac{1}{\psi(n+1) - \ln n} - 2n \leq a.$$

Define for $x > 0$

$$\phi(x) = \frac{1}{\psi(x+1) - \ln x} - 2x.$$

Differentiating ϕ and utilizing (8) and (9) reveals that for $x > \frac{12}{5}$

$$(\psi(x+1) - \ln x)^2 \phi'(x) = \frac{1}{x} - \psi'(x+1) - 2(\psi(x+1) - \ln x)^2$$
$$< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2 = \frac{12 - 5x}{360x^5} < 0,$$

and then the function ϕ strictly decreases with $x > \frac{12}{5}$.

Straightforward calculation produces

$$\phi(1) = \frac{1}{1-\gamma} - 2 = 0.36527211862544155\cdots,$$

$$\phi(2) = \frac{1}{\frac{3}{2} - \gamma - \ln 2} - 4 = 0.35469600731465752\cdots,$$

$$\phi(3) = \frac{1}{\frac{11}{6} - \gamma - \ln 3} - 6 = 0.34898948531361115\cdots.$$

Therefore, the sequence

$$\phi(n) = \frac{1}{\psi(n+1) - \ln n} - 2n, \qquad n \in N$$

is strictly decreasing. This leads to

$$\lim_{n\to\infty} \phi(n) < \phi(n) \leq \phi(1) = \frac{1}{1-\gamma} - 2.$$

Making use of asymptotic formula of ψ (see [1, p. 259])

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \qquad (x \to \infty),$$

we conclude that

$$\lim_{n\to\infty} \phi(n) = \lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{\frac{1}{3} + O(x^{-2})}{1 + O(x^{-1})} = \frac{1}{3}.$$

This completes the proof of the theorem 2.

References

[1] M. Abramowitz and I.Stegun (Eds), Handbook of Mathematical Function with Formulas, Graphs, and Mathematical Tables, 4th printing, with corrections, Applied Mathematics Series 55, National Bureau of Standards, Washington, 1965.

[2] M. Merkle, Logarithmic convexity inequalities for the gemma function, J. Math. Anal. Appl, **203**(1996), 369–380.

[3] G. Pólya and G. Szegö, Problems and Theorems in analysis, Vol.I and II, Springer-Verlag, Berlin, Heidelberg, 1972.

[4] Z. Sasvári, Inequalities for binomial coefficients, J. Math. Anal. Appl. **236**(1999), 223-226.

[5] L. Tóth, E 3432, Amer. Math. Monthly **98**(1991), 264; **99**(1992), 684-685.

[6] Zh-x.Wang and D.-R.Guo, Introduction to Special Function, the Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000(Chinese).