ON THE INFERIOR AND SUPERIOR k-TH POWER PART OF A POSITIVE INTEGER AND DIVISOR FUNCTION

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ABSTRACT: For any positive integer n, let $a(n)$ and $b(n)$ denote the inferior and superior k-th power part of n respectively. That is, $a(n)$ denotes the largest k-th power less than or equal to n, and $b(n)$ denotes the smallest k-th power greater than or equal to n. In this paper, we study the properties of the sequences $\{a(n)\}$ and $\{b(n)\}$, and give two interesting asymptotic formulas.

Key words and phrases: Inferior and superior k-th power part; Mean value; Asymptotic formula.

1. INTRODUCTION

For a fixed positive integer $k>1$, and any positive integer n, let $a(n)$ and $b(n)$ denote the inferior and superior k-th power part of n respectively .That is, $a(n)$ denotes the largest k-th power less than or equal to n, $b(n)$ denotes the smallest k-th power greater than or equal to n. For example, let k=2 then $a(1)=a(2)=a(3)=1, a(4)=a(5)=\cdots=a(7)=4, \cdots, b(1)=1, b(2)=b(3)=b(4)=4, b(5)=b(6)=\cdots=b(8)=8\cdots$; let k=3 then $a(1)=a(2)=\cdots=a(7)=1$, $a(8)=a(9)=\cdots=a(26)=8,\cdots,b(1)=1$, $b(2)=b(3)=\cdots=b(8)=8$, $b(9)=b(10)=\cdots=b(27)=27\cdots$. In problem 40 and 41 of [1], Professor F. Smarandache asks us to study the properties of the sequences $\{a(n)\}$ and $\{b(n)\}$. About these problems, Professor Zhang Wenpeng [4] gave two interesting asymptotic formulas of the cure part of a positive integer. In this paper, we give asymptotic formulas of the k-th power part of a positive integer. That is, we shall prove the following:

Theorem 1. For any real number $x>1$, we have the asymptotic formula

$$\sum_{n\le x} d(a(n))=\frac{1}{kk!}\left(\frac{6}{k\pi^2}\right)^{k-1}A_0x\ln^k x+A_1x\ln^{k-1}x+\cdots+A_{k-1}x\ln x+A_kx+O(x^{1-\frac{1}{2k}+\varepsilon})$$

where A_0, A_1, $\cdots$ A_k are constants, especially when k equals to 2, $A_0=1$; $d(n)$ denotes the Dirichlet divisor function, ε is any fixed positive number.

For the sequence $\{b(n)\}$, we can also get similar result.

Theorem 2. For any real number $x>1$, we have the asymptotic formula

$$\sum_{n\le x} d(b(n))=\frac{1}{kk!}\left(\frac{6}{k\pi^2}\right)^{k-1}A_0x\ln^k x+A_1x\ln^{k-1}x+\cdots+A_{k-1}x\ln x+A_kx+O(x^{1-\frac{1}{2k}+\varepsilon})$$

2. A SIMPLE LEMMA

To complete the proof of the theorems, we need following

Lemma 1. For any real number $x>1$, we have the asymptotic formula

$$\sum_{n\le x} d(n^k)=\frac{1}{k!}(\frac{6}{\pi^2})^{k-1}B_0 x\ln^k x+B_1 x\ln^{k-1}x+\cdots+B_{k-1}x\ln x+B_k x+O(x^{\frac{1}{2}+\varepsilon}).$$

where B_0, B_1, $\cdots$ B_k are constants, especially when $k=2$, $A_0=1$; ε is any fixed positive number.

Proof. Let $s=\sigma+it$ be a complex number and $f(s)=\sum_{n=1}^{\infty}\frac{d(n^k)}{n^s}$.

Note that $d(n^k)<<n^{\varepsilon}$, So it is clear that $f(s)$ is a Dirichlet series absolutely convergent in Re(s)>1, by the Euler Product formula [2] and the definition of $d(n)$ we have

$$f(s)=\prod_p\left(1+\frac{d(p^k)}{p^s}+\frac{d(p^{2k})}{p^{2s}}+\cdots+\frac{d(p^{kn})}{p^{ns}}+\cdots\right)$$

$$=\prod_p\left(1+\frac{k+1}{p^s}+\frac{2k+1}{p^{2s}}+\cdots+\frac{kn+1}{p^{ns}}+\cdots\right)$$

$$=\zeta^2(s)\prod_p\left(1+(k-1)\frac{1}{p^s}\right)$$

$$=\zeta^2(s)\prod_p\left((1+\frac{1}{p^s})^{k-1}-C_{k-1}^2\frac{1}{p^{2s}}-\cdots-\frac{1}{p^{(k-1)s}}\right)$$

$$=\frac{\zeta^{k+1}(s)}{\zeta^{k-1}(2s)}g(s). \tag{1}$$

where $\zeta(s)$ is Riemann zeta-function and $\prod_p$ denotes the product over all primes.

From (1) and Perron's formula [3] we have

$$\sum_{n\le x} d(n^k)=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{\zeta^{k+1}(s)}{\zeta^{k-1}(2s)}g(s)\frac{x^s}{s}ds+O\left(\frac{x^{2+\varepsilon}}{T}\right), \tag{2}$$

where $g(s)$ is absolutely convergent in $\mathrm{Re}(s)>\frac{1}{2}+\varepsilon$. We move the integration in (2) to $\mathrm{Re}(s)=\frac{1}{2}+\varepsilon$. The pole at $s=1$ contributes to

$$\frac{1}{k!}(\frac{6}{\pi^2})^{k-1}B_0 x\ln^k x+B_1 x\ln^{k-1}x+\cdots+B_{k-1}x\ln x+B_k x, \tag{3}$$

where $B_0, B_1, \ldots B_k$ are constants, especially when $k=2, B_0=1$.

For $\frac{1}{2}\le\sigma<1$, note that $\zeta(s)=\zeta(\sigma+it)\le|t|^{\frac{1-\sigma}{2}+\varepsilon}$. Thus, the horizontal integral contributes to

$$O\left(x^{\frac{1}{2}+\varepsilon}+\frac{x^2}{T}\right), \tag{4}$$

and the vertical integral contributes to

$$O\left(x^{\frac{1}{2}+\varepsilon}\ln^4 T\right). \tag{5}$$

On the line $\mathrm{Re}(s)=\frac{1}{2}+\varepsilon$, taking parameter $T=x^{\frac{3}{2}}$, then combining (2), (3), (4) and (5) we have

$$\sum_{n\le x} d(n^k)=\frac{1}{k!}\left(\frac{6}{\pi^2}\right)^{k-1} B_0 x\ln^k x+B_1 x\ln^{k-1}x+\cdots+B_k x+O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

This proves Lemma 1.

3. PROOFS OF THE THEOREMS

Now we complete the proof of the Theorems. First we prove Theorem 1.

For any real number $x>1$, Let M be a fixed positive integer such that

$$M^k \le x<(M+1)^k, \tag{6}$$

then, from the definition of $a(n)$, we have

$$\begin{aligned}
\sum_{n\le x} d(a(n)) &= \sum_{m=2}^{M}\ \sum_{(m-1)^k\le n<m^k} d(a(n)+\sum_{M^k\le n\le x} d(a(n))\\
&=\sum_{m=1}^{M-1}\ \sum_{m^k\le n<(m+1)^k} d(m^k)+\sum_{M^k\le n\le x} d(M^k)\\
&=\sum_{m=1}^{M-1}(C_k^1 m^{k-1}+C_k^2 m^{k-2}+\cdots+1)d(m^k)+O\left(\sum_{M^k\le n\le (M+1)^k} d(M^k)\right),\\
&=k\sum_{m=1}^{M} m^{k-1}d(m^k)+O(M^{k-1+\varepsilon}),
\end{aligned} \tag{7}$$

where we have used the estimate $d(n)<<n^\varepsilon$.

Let $B(y)=\sum_{n\le y} d(n^k)$, then by Abel's identity and Lemma 1, we have

$$\sum_{m=1}^{M} m^{k-1}d(m^k)=M^{k-1}B(M)-(k-1)\int^{M} y^{k-2}B(y)dy+O(1)$$

$$=M^{k-1}\left(\frac{1}{k!}(\frac{6}{\pi^2})^{k-1}B_0 M\ln^k M+B_1 M\ln^{k-1}M+\cdots+B_k M\right)$$

$$-(k-1)\int^{M}\left(\frac{1}{k!}(\frac{6}{\pi^2})^{k-1}B_0 y^{k-1}\ln^k y + B_1 y^{k-1}\ln^{k-1} y + \cdots + B_k y^{k-1}\right)dy$$

$$+O\left(M^{k-\frac{1}{2}+\varepsilon}\right)$$

$$=\frac{1}{kk!}\left(\frac{6}{\pi^2}\right)^{k-1} B_0 M^k \ln^k M + C_1 M^k \ln^{k-1} M + \cdots + C_{k-1}M^k + O\left(M^{k-\frac{1}{2}+\varepsilon}\right). \quad (8)$$

Applying (7) and (8) we obtain the asymptotic formula

$$\sum_{n\le x} d(a(n)) = \frac{1}{k!}\left(\frac{6}{\pi^2}\right)^{k-1} B_0 M^k \ln^k M + C_1 M^k \ln^{k-1} M + \cdots + C_{k-1}M^k + O\left(M^{k-\frac{1}{2}+\varepsilon}\right), \quad (9)$$

where $B_0, C_1, \cdots, C_{k-1}$ are constants.

From (6) we have the estimates

$$0 \le x - M^k < (M+1)^k - M^k = kM^{k-1} + C_k^2 M^{k-2} + \cdots + 1$$

$$= M^{k-1}(k + C_k^2 \frac{1}{M} + \cdots + \frac{1}{M^{k-1}}) << x^{\frac{k-1}{k}}, \quad (10)$$

and

$$\ln^k x = k^k \ln^k M + O\left(\frac{\ln^{k-1} x}{x^{\frac{1}{k}}}\right) = k^k \ln^k M + O(x^{-\frac{1}{k}+\varepsilon}). \quad (11)$$

Combining (9), (10) and (11) we have

$$\sum_{n\le x} d(a(n)) = \frac{1}{kk!}(\frac{6}{k\pi^2})^{k-1} A_0 x \ln^k x + A_1 x \ln^{k-1} x + \cdots + A_{k-1} x \ln x + A_k x + O(x^{1-\frac{1}{2k}+\varepsilon}),$$

where A_0 equals to B_0.

This proves Theorem 1.

Using the methods of proving Theorem 1 we can also prove Theorem 2. This completes the proof of the Theorems.

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