The equations $m \cdot S(m) = n \cdot S(n)$ and $m \cdot S(n) = n \cdot S(m)$ have infinitily many solutions

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Let be $S : N^* \to N^*$ the Smarandache function.

$$S(n) = \min \{ k \mid n \leq_d k! \}$$

where $\leq_d$ is the order generated by:

$$\underset{d}{\wedge} = \text{g.c.d.}$$

$$\overset{d}{\vee} = \text{s.c.m.}$$

on set N^*.

It is known that $\mathcal{N}_d = (N^*, \underset{d}{\wedge}, \overset{d}{\vee})$ is a lattice where 1 is the smallest element and 0 is the biggest element. The order $\leq_d$ is defined like in any lattice by:

$$n \leq_d m \Leftrightarrow n \underset{d}{\wedge} m = n \Leftrightarrow n \overset{d}{\vee} m = m$$

or, in other terms:

$$n \leq_d m \Leftrightarrow n | m .$$

Next we will study two diophantine equations which contain the Smarandache function.

Reminding of two of the features of Smarandache's function which we will need further:

1. Smarandache's function satisfies:

$$S(m \overset{d}{\vee} n) = \max\{S(m), S(n)\}$$

2. To calculate $S(p^\alpha)$:

   2.a. we will write the exponent in the generalized base $[p]$ definite by the sequence with general term:

$$a_i(p)=\frac{p^i-1}{p-1}$$

who satisfies:

$$a_{i+1}(p)=p\cdot a_i(p)+1$$

that is:

$$[p]:\ a_1(p), a_2(p), \ldots$$

2.b. the result is read in the standard base (p) definite by the sequence:

$$b_i(p)=p^i$$

who satisfies:

$$b_{i+1}(p)=p\cdot b_i(p)$$

that is:

$$(p):\ 1, p, p^2, p^3, \ldots$$

2.c. the number obtained will be multiply by p.

Proposition: The equation

$$mS(m)=nS(n) \tag{1}$$

has infinity many solutions in the next two cases:

1. $m=n$ - obvious
2. $m>n$ with $m=d\cdot a$, $n=d\cdot b$ satisfying $m\underset{d}{\wedge}n=d$, $d\underset{d}{\wedge}a=1$, $d\underset{d}{\wedge}b>1$ and the dual of this condition for $m<n$.

The equation

$$mS(n)=nS(m) \tag{2}$$

has infinity many solutions in the next two cases:

1. $m=n$ - obvious
2. $m>n$ and $m\underset{d}{\wedge}n=1$

Proof

Let's consider $m>n$. We distinguish the next cases:

1. $m\underset{d}{\wedge}n=1$ that is $(m,n)=1$.

Then from equation (1) we can deduce: $m\leq_d S(n)$; then $m\leq S(n)$. But $S(n)\leq n$ for every n and as $n<m$ we get the contradiction: $S(n)<m$.

For the equation (2) we have: $m \leq_d S(m) \Rightarrow m \leq S(m) \Rightarrow m = S(m) \Rightarrow m = 4$ or m is a prime number. If $m = 4$ the equation becomes:

$$4 \cdot S(n) = 4 \cdot n \;\Rightarrow\; n = S(n) \;\Rightarrow\; n = 4 \text{ or } n \text{ is a prime number}$$

So in this case the equation has for solutions the pairs of numbers:

$$(4,4)\,,\ (4,p)\,,\ (p,4)\,,\ (p,q) \text{ with } p,q \text{ prime numbers.}$$

2. If $m \underset{d}{\wedge} n = d \neq 1$, so:

$$\begin{cases} m = d \cdot a \\ n = d \cdot b \end{cases}, \text{ cu } a \underset{d}{\wedge} b = 1 \qquad (3)$$

the equation (1) becomes:

$$a \cdot S(m) = b \cdot S(n) \qquad (4)$$

From condition $m > n$ we deduce:

$$a > b$$

We can distinguish the next possibilities:

a) $d \underset{d}{\wedge} a = 1,\ d \underset{d}{\wedge} b = 1$

If we note:

$$\mu = S(m)\,,\ \nu = S(n)$$

we have:

$$\begin{aligned} \mu = S(m) = S(d \cdot a) = S(d \overset{d}{\vee} a) = \max(S(d), S(a)) \\ \nu = S(n) = S(d \cdot b) = S(d \overset{d}{\vee} b) = \max(S(d), S(b)) \end{aligned} \qquad (5)$$

and the equation (1) is equivalent with:

$$\frac{m}{n} = \frac{S(n)}{S(m)} \;\Leftrightarrow\; \frac{a}{b} = \frac{\nu}{\mu} \qquad (6)$$

From (5) we deduce for μ and ν the possibilities:

a1) $\mu = S(d)\,,\ \nu = S(d)$, that is:

$$S(d) \geq S(a) \text{ and } S(d) \geq S(b)$$

In this case (6) becomes:

$$\frac{a}{b} = 1 \text{ - false}$$

a2) $\mu = S(d)\,,\ \nu = S(b)$, that is:

$$S(d) \geq S(a) \text{ and } S(d) < S(b)$$

In this case (6) becomes:

$$\frac{a}{b} = \frac{S(b)}{S(d)} \;\Rightarrow\; aS(d) = bS(b)$$

But $a \underset{d}{\wedge} b = 1$, so we must have:

$$a \leq_d S(b) \text{ so } a \leq S(b) \tag{7}$$

and in the same time:

$$S(b) \leq b < a \text{ - contradicts (7)}$$

a3) $\mu = S(a)$, $\nu = S(d)$ that is:

$$S(a) > S(d) \text{ and } S(d) \geq S(b) \tag{8}$$

In this case the equation (6) is:

$$\frac{a}{b} = \frac{S(d)}{S(a)}$$

that is:

$$aS(a) = bS(d) \tag{9}$$

Then from $a \underset{d}{\wedge} b = 1 \Rightarrow a \leq_d S(d)$ and $b \leq_d S(a)$. So:

$$S(a) \leq a \leq S(d) \text{ - contradicts (8)}$$

a4) $\mu = S(a)$, $\nu = S(b)$

In this case the equation (6) becomes:

$$\frac{a}{b} = \frac{S(b)}{S(a)} \text{ with } a \underset{d}{\wedge} b = 1$$

and we are in the case 1.

For the equation (2) which can be also write:

$$aS(n) = bS(m) \tag{10}$$

that is: $a\nu = b\mu$

- in the conditions **a1)** it becomes:

$$a = b \text{ - false}$$

- in the conditions **a2)** it becomes:

$$aS(b) = bS(d)$$

and as $a \underset{d}{\wedge} b = 1$ we deduce:

$$a \leq_d S(d), \ b \leq_d S(b).$$

So $b \leq S(b)$, that is $b = S(b)$, so $b = 4$ or $b = p$ - prime number

and the equation becomes:

$$a = S(d)$$

and as $S(d) \underset{d}{\wedge} d > 1$ we obtain the contradiction:

$$a \underset{d}{\wedge} d > 1$$

- in the conditions **a3)** it becomes:

$$aS(d) = bS(a)$$

and because $a \underset{d}{\wedge} b = 1$ we must have $a \leq_d S(a)$ that is $a = S(a)$.

So the equation is:

$$S(d) = b$$

As $d \underset{d}{\wedge} S(d) > 1$ it results $d \underset{d}{\wedge} b > 1$ - false.

- in the conditions **a4)** the equation becomes:

$$aS(b) = bS(a)$$

that is the equation (2) in the case 1.

b) $d \underset{d}{\wedge} a > 1$ and $d \underset{d}{\wedge} b = 1$

As (1) is equivalent with (4) from $a \underset{d}{\wedge} b = 1$ it results:

$$a \leq_d S(n) \text{ and } b \leq_d S(m)$$

From the hypothesis ($d \underset{d}{\wedge} a > 1$) it results:

$$S(m) = S(a \cdot d) \geq \max\{S(a), S(d)\} \quad (11)$$

If in (11) the inequality is not top, that is:

$$S(m) = \max\{S(d), S(a)\}$$

and as

$$S(n) = \max\{S(d), S(b)\} \quad (12)$$

we are in the in the case a). Let's suppose that in (11) the inequality is top:

$$S(m) > \max\{S(a), S(d)\}$$

It results:

$$S(m) > S(a) \quad (13)$$

$$S(m) > S(d) \quad (14)$$

Reminding of (11) we have the next cases:

b1) $S(n) = S(d)$

The equation(4) becomes:

$$aS(m) = bS(d)$$

and from $a > b$ it results $S(d) > S(m)$ - false (13).

b2) $S(n) = S(b)$

The equation (4) becomes:

$$aS(m) = bS(b)$$

As $\gcd(a, b) = 1$ it results $a \leq_d S(b)$ so $a \leq S(b)$ - false because $S(b) \leq b < a$.

c) $d \underset{d}{\wedge} a = 1$ and $d \underset{d}{\wedge} b > 1$

We get:

$$S(m) = S(d \cdot a) = S(d \overset{d}{\vee} a) = \max\{S(d), S(a)\}$$
$$S(n) = S(d \cdot b) \geq \max\{S(d), S(b)\}$$

If the last inequality is not top, we have the case a). So let it be:

$$S(n) > \max\{S(d), S(b)\},$$

that is:

$$S(n) > S(d) \tag{15}$$

and

$$S(n) > S(b) \tag{16}$$

c1) $S(m) = S(d)$, that is $S(d) \geq S(a)$. The equation becomes:

$$aS(d) = bS(n)$$

We can't get a contradiction and we can see that the equation has solutions like this:

$$m = p^{\alpha} \cdot a$$
$$n = p^{\alpha + x}$$

So $b = p^x, d = p^{\alpha}$. The condition $a > b$ becomes $a > p^x$. We must have also $a \underset{d}{\wedge} p^{\alpha} = 1$, that is $a \underset{d}{\wedge} p = 1$.

The equation becomes:

$$p^{\alpha} a \cdot S(p^{\alpha}) = p^{\alpha+x} S \cdot (p^{\alpha+x})$$

It results:

$$a = \frac{p^x S(p^{\alpha+x})}{S(p^{\alpha})} = \frac{p^x p((\alpha+x)_{[p]})_{(p)}}{p(\alpha_{[p]})_{(p)}} = \frac{p^x((\alpha+x)_{[p]})_{(p)}}{(\alpha_{[p]})_{(p)}}.$$

We can see that choosing α this way:

$$(\alpha_{[p]})_{(p)} = p^x = (1\underbrace{00\ldots0}_{x\ times})_{(p)} \Rightarrow \alpha = \alpha_{[p]} = (1\underbrace{00\ldots0}_{x\ times})_{[p]} = a_{x+1}(p)$$

we get:

$$a = ((\alpha + x)_{[p]})_{(p)} \in N$$

We must also put the condition $a \underset{d}{\wedge} p = 1$ which we can get choosing convenient values for x.

Example: For $n = 3$ we have:

(3): 1, 3, 9, 27, ...

[3]: 1, 4, 13, 40, 121, ...

Considering $x = 2$ we get (from condition $(\alpha_{[p]})_{(p)} = p^x$):

$$(\alpha_{[3]})_{(3)} = 3^x = 3^2 = 100_{(3)} \Rightarrow \alpha = 100_{[3]} = 13 \Rightarrow$$

$$a = S(p^{\alpha+x}) = S(3^{13+2}) = S(3^{15}) = (15_{[3]})_{(3)} = 102_{(3)} = 11$$

So, $(m = 3^{13} \cdot 11, n = 3^{15})$ is solution for equation (1).

Equation (2) which has the form:

$$aS(n) = bS(d)$$

has no solutions because from $a > b \Rightarrow S(d) > S(n)$ - false.

References:

1. C.Dumitrescu, V.Seleacu **The Smarandache Function** *Erhus University Press 1996*
2. Department of Mathematics, University of Craiova **Smarandache Notions Journal** *vol. 7, no. 1-2-3, august 1996*