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An infinity series involving the Smarandache-type function

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Abstract In this paper, we using the elementary method to study the convergent property of one class Dirichlet series involving a special sequences, and give several interesting identities for it.

Keywords Riemann zeta-function, infinity series, identity.

§1. Introduction and Results

For any positive integer n and $m \geq 2$, the Smarandache-type function $B_m(n)$ is defined as the largest m-th power dividing n. That is,

$$B_m(n) = \max\{x^m : x^m \mid n\}(\forall n \in N^*).$$

For example, $B_2(1) = 1$, $B_2(2) = 1$, $B_2(3) = 1$, $B_2(4) = 2$, $B_2(5) = 1$, $B_2(6) = 1$, $B_2(7) = 1$, $B_2(8) = 2$, $B_2(9) = 3$, $\cdots$. This function was first introduced by Professor Smarandache. In [1], Henry Bottomley presented that $B_m(n)$ is a multiplicative function. That is,

$$(\forall a, b \in N)(a, b) = 1 \Rightarrow B_m(a \cdot b) = B_m(a) \cdot B_m(b).$$

It is easily to show that $B_m(p^\alpha) = p^{im}$, $\alpha = im + l$, $(i \geq 0, 0 \leq l < m)$, where p is a prime. So, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime powers decomposition of n, then the following identity is obviously:

$$B_m(n) = B_m(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{i_1 m} p_2^{i_2 m} \cdots p_r^{i_r m},$$

where $\alpha_j = i_j m + l_j$ $(i_j \geq 0, 0 \leq l_j < m)$.

Similarly, for any positive integer n and any fixed positive integer m, we define another Smarandache function $C_m(n)$ as following:

$$C_m(n) = \max\{x \in N : x^m \mid n\}(\forall n \in N^*).$$

Obviously, $C_m(n)$ is also a multiplicative function.

From the definition of $B_m(n)$ and $C_m(n)$, we may immediately get

$$B_m(n) = C_m(n)^m.$$

Now let k be a fixed positive integer, for any positive integer n, we define the arithmetical function $\delta_k(n)$ as following:

$$\delta_k(n) = \max\{d : d \mid n, (d,k) = 1\}.$$

For example, $\delta_2(1) = 1$, $\delta_2(2) = 1$, $\delta_2(3) = 1$, $\delta_2(4) = 1$, $\delta_3(6) = 2, \cdots$. About the elementary properties of this function, many scholars have studied it and got some useful results (see reference [2], [3]). In reference [2], Xu Zhefeng studied the divisibility of $\delta_k(n)$ by $\varphi(n)$, and proved that $\varphi(n) \mid \delta_k(n)$ if and only if $n = 2^\alpha 3^\beta$, where $\alpha > 0$, $\beta \geq 0$, $\alpha, \beta \in N$. In reference [3], Liu Yanni and Gao Peng studied the mean value properties of $\delta_k(b_m(n))$, and obtained an interesting mean value formula for it. That is, they proved the following conclusion:

Let k and m are two fixed positive integers. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} \delta_k(b_m(n)) = \frac{x^2}{2} \frac{\zeta(2m)}{\zeta(m)} \prod_{p|k} \frac{p^m + 1}{p^{m-1}(p+1)} + O(x^{\frac{3}{2}+\epsilon}),$$

where ϵ denotes any fixed positive number, $\zeta(s)$ is the Riemann zeta-function, and $\prod_{p|k}$ denotes the product over all different prime divisors of k.

Let $\mathcal{A}$ denotes the set of all positive integers n satisfying the equation $B_m(n) = \delta_k(n)$. That is, $\mathcal{A} = \{n \in N, B_m(n) = \delta_k(n)\}$. In this paper, we using the elementary method to study the convergent property of the Dirichlet series involving the set $\mathcal{A}$, and give several interesting identities for it. That is, we shall prove the following conclusions:

Theorem 1. Let $m \geq 2$ be a fixed positive integer. Then for any real number $s > 1$, we have the identity:

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^s} = \zeta(ms) \prod_{p|k} \frac{(1 - \frac{1}{p^{ms}})^2}{1 - \frac{1}{p^s}}.$$

Theorem 2. For any complex number s with $Re(s) > 2$, we have the identity:

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{B_m(n)}{n^s} = \zeta(ms-1) \prod_{p|k} \frac{(1 - \frac{1}{p^{ms}})^2}{1 - \frac{1}{p^s}},$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes.

§2. Proof of the theorems

Now we complete the proof of our Theorems. First we define the arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & \text{if } \ n \in \mathcal{A}, \\ 0, & \text{if } \ otherwise. \end{cases}$$

For any real number $s > 0$, it is clear that

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and $\sum\limits_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if $s > 1$, thus $\sum\limits_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s}$ is also convergent if $s > 1$. Now we find the set $\mathcal{A}$. From the definition of $B_m(n)$ and $\delta_k(n)$ we know that $B_m(n)$ and $\delta_k(n)$ both are multiplicative functions. So in order to find all solutions of the equation $B_m(n) = \delta_k(n)$, we only discuss the case $n = p^\alpha$. Let $\alpha = im + l$, where $i \geq 0$, $0 \leq l < m$, then $B_m(n) = p^{im}$. If $n = p^\alpha$, $(p, k) = 1$, now $\delta_k(n) = p^\alpha$, then the equation $B_m(n) = \delta_k(n)$ has solution if and only if $\alpha = im$, $i \geq 0$. If $n = p^\alpha$, $p \mid k$, now $\delta_k(n) = 1$, then the equation $B_m(n) = \delta_k(n)$ has solution if and only if $\alpha = l$, $0 \leq l < m$.

Thus, by the Euler product formula, we have

$$\begin{aligned}
\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} &= \prod_{p} \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots + \frac{a(p^{m-1})}{p^{(m-1)s}} + \cdots\right) \\
&= \prod_{p|k} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(m-1)s}}\right) \prod_{p\dagger k} \left(1 + \frac{1}{p^{ms}} + \frac{1}{p^{2ms}} + \cdots\right) \\
&= \prod_{p|k} \frac{1 - \frac{1}{p^{ms}}}{1 - \frac{1}{p^s}} \prod_{p\dagger k} \frac{1}{1 - \frac{1}{p^{ms}}} \\
&= \prod_{p} \frac{1}{1 - \frac{1}{p^{ms}}} \prod_{p|k} \frac{(1 - \frac{1}{p^{ms}})^2}{1 - \frac{1}{p^s}} \\
&= \zeta(ms) \prod_{p|k} \frac{(1 - \frac{1}{p^{ms}})^2}{1 - \frac{1}{p^s}}.
\end{aligned}$$

This completes the proof of Theorem 1.

Now we come to prove Theorem 2. Let $s = \sigma + it$ be a complex number. Note that $B_m(n) \ll n$, so it is clear that $\sum\limits_{n=1}^{\infty} \frac{B_m(n)}{n^s}$ is an absolutely convergent series for $Re(s) > 2$, by the Euler product formula and the definition of $B_m(n)$ we get

$$\begin{aligned}
\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty}\frac{B_m(n)}{n^s} &= \prod_{p}\left(1+\frac{B_m(p)}{p^s}+\frac{B_m(p^2)}{p^{2s}}+\cdots\right)\\
&= \prod_{p|k}\left(1+\frac{B_m(p)}{p^s}+\frac{B_m(p^2)}{p^{2s}}+\cdots+\frac{B_m(p^{m-1})}{p^{(m-1)s}}\right)\\
&\quad \prod_{p\dagger k}\left(1+\frac{B_m(p^m)}{p^{ms}}+\frac{B_m(p^{2m})}{p^{2ms}}+\cdots\right)\\
&= \prod_{p|k}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{(m-1)s}}\right)\prod_{p\dagger k}\left(1+\frac{p^m}{p^{ms}}+\frac{p^{2m}}{p^{2ms}}+\cdots\right)\\
&= \prod_{p|k}\frac{1-\frac{1}{p^{ms}}}{1-\frac{1}{p^s}}\prod_{p\dagger k}\frac{1}{1-\frac{1}{p^{ms-1}}}\\
&= \prod_{p}\frac{1}{1-\frac{1}{p^{ms-1}}}\prod_{p|k}\frac{(1-\frac{1}{p^{ms}})^2}{1-\frac{1}{p^s}}\\
&= \zeta(ms-1)\prod_{p|k}\frac{(1-\frac{1}{p^{ms}})^2}{1-\frac{1}{p^s}}.
\end{aligned}$$

This completes the proof of Theorem 2.

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