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An equation involving the F.Smarandache multiplicative function[1]

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Abstract For any positive integer n, we call an arithmetical function $f(n)$ as the F.Smarandache multiplicative function if $f(1) = 1$, and if $n > 1$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, then $f(n) = \max\limits_{1 \leq i \leq k} \{f(p_i^{\alpha_i})\}$. The main purpose of this paper is using the elementary methods to study the solutions of an equation involving the F.Smarandache multiplicative function, and give its all positive integer solutions.

Keywords F.Smarandache multiplicative function, function equation, positive integer solution, elementary methods.

§1. Introduction and result

For any positive integer n, we call an arithmetical function $f(n)$ as the F.Smarandache multiplicative function if $f(1) = 1$, and if $n > 1$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, then $f(n) = \max\limits_{1 \leq i \leq k} \{f(p_i^{\alpha_i})\}$. For example, the function $S(n) = \min\{m : m \in N,\ n|m!\}$ is a F.Smarandache multiplicative function. From the definition of $S(n)$, it is easy to see that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, we have

$$S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}.$$

So we can say that $S(n)$ is a F.Smarandache multiplicative function. In fact, this function be the famous F.Smarandache function, the first few values of it are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots\cdots$. About the arithmetical properties of $S(n)$, some authors had studied it, and obtained some valuable results. For example, Farris Mark and Mitchell Patrick [2] studied the upper and lower bound of $S(p^\alpha)$, and proved that

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Professor Wang Yongxing [3] studied the mean value properties of $S(n)$, and obtained a sharper asymptotic formula, that is

$$\sum_{n \leq x} S(n) = \frac{\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

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Lu Yaming [4] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite groups positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [5] proved for any positive integer $k \geq 2$, there exist infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$ satisfied the following inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

In [6], Fu Jing proved more general conclusion. That is, if the positive integer k and m satisfied the one of the following conditions:

(a) $k > 2$ and $m \geq 1$ are all odd numbers.

(b) $k \geq 5$ is odd, $m \geq 2$ is even.

(c) Any even numbers $k \geq 4$ and any positive integer m;

then the equation

$$m \cdot S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite groups of positive integer solutions $(m_1, m_2, \cdots, m_k)$.

In [7], Xu Zhefeng studied the value distribution of $S(n)$, and obtained a deeply result. That is, he proved the following Theorem:

Let $P(n)$ be the largest prime factor of n, then for any real numbers $x > 1$, we have the asymptotic formula:

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function.

On the other hand, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, we define

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_k^{\alpha_k}\}.$$

Obviously, this function is also a Smarandache multiplicativa function, which is called F.Smarandache LCM function. About the properties of this function, there are many scholars have studied it, see references [8] and [9].

Now, we define another arithmetical function $\overline{S}(n)$ as follows: $\overline{S}(1) = 1$, when $n > 1$ and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the fractorization of n into prime powers, then we define

$$\overline{S}(n) = \max\{\alpha_1 p_1,\ \alpha_2 p_2,\ \alpha_3 p_3,\ \cdots,\ \alpha_k p_k\}.$$

It is easy to prove that this function is also a F.Smarandache multiplicative function. About its elementary properties, we know very little, there are only some simple properties mentioned in [7]. That is, if we replace $S(n)$ with $\overline{S}(n)$ in (1), it is also true.

The main purpose of this paper is using the elementary methods to study the solutions of an equation involving $\overline{S}(n)$. That is, we shall study all positive integer solutions of the equation

$$\sum_{d|n} \overline{S}(d) = n, \tag{2}$$

where $\sum_{d|n}$ denotes the summation over all positive factors of n.

Obviously, there exist infinite positive integer n, such that $\sum_{d|n} \overline{S}(d) > n$. For example, let $n = p$ be a prime, then $\sum_{d|n} \overline{S}(d) = 1 + p > p$. At the same time, there are also infinite positive integer n, such that $\sum_{d|n} \overline{S}(d) < n$.

In fact, let $n = pq$, p and q are two different odd primes with $p < q$, then we have $\sum_{d|n} \overline{S}(d) = 1 + p + 2q < pq$. So a natural problem is whether there exist infinite positive integer n satisfying (2)? We have solved this problem completely in this paper, and proved the following conclusion:

Theorem. For any positive integer n, the equation (2) holds if and only if $n = 1,\ 28$.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Firstly, we prove some special cases:

(i) If $n = 1$, $\sum_{d|n} \overline{S}(d) = \overline{S}(1) = 1$, then $n = 1$ is a solution of equation (2).

(ii) If $n = p^\alpha$ is the prime powers, then (2) doesn't hold.

In fact, if (2) holds, then from the definition of $\overline{S}(n)$, we have

$$\sum_{d|n} \overline{S}(d) = \sum_{d|p^\alpha} \overline{S}(d) = 1 + p + 2p + \cdots + \alpha p = p^\alpha. \tag{3}$$

Obviously, the right side of (3) is a multiple of p, but the left side is not divided by p, a contradiction. So if n is a prime powers, (2) doesn't hold.

(iii) If $n > 1$ and the least prime factor powers of n is 1, then the equation (2) also doesn't hold. Now, if $n = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k} = p_1 n_1$ satisfied (2), then from the conclusion (ii), we know that $k \geq 2$, so from the definition of $\overline{S}(n)$, we have

$$\sum_{d|n} \overline{S}(d) = \sum_{d|n_1} \overline{S}(d) + \sum_{d|n_1} \overline{S}(p_1 d) = 2\sum_{d|n_1} \overline{S}(d) + p_1 - 1 = p_1 n_1. \tag{4}$$

Obviously, two sides of (4) has the different parity, it is impossible.

We get immediately from the conclusion (iii), if n is a square-free number, then n can't satisfy (2).

Now we prove the general case. Provided integer $n > 1$ satisfied equation (2), from (ii) and (iii), we know that n has two different prime powers at least, and the least prime factor power of n is larger than 1. So we let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $\alpha_1 > 1$, $k \geq 2$. Let $\overline{S}(n) = \alpha p$, we discuss it in the following cases:

(A) $\alpha = 1$. Then p must be the largest prime factors of n, let $n = n_1 p$, note that, if $d|n_1$, we have$\overline{S}(d) \leq p - 1$, so from $\sum\limits_{d|n} \overline{S}(d) = n$ we get

$$\begin{aligned} n_1 p &= n = \sum_{d|n_1 p} \overline{S}(d) = \sum_{d|n_1} \overline{S}(d) + \sum_{d|n_1} \overline{S}(dp) \\ &= \sum_{d|n_1} \overline{S}(d) + \sum_{d|n_1} p \leq 1 + \sum_{\substack{d|n_1 \\ d>1}} (p-1) + pd(n_1) \\ &= 2 + (2p-1)d(n_1) - p \end{aligned} \tag{5}$$

or

$$n_1 + 1 < 2d(n_1), \tag{6}$$

where $d(n_1)$ is the Direchlet divisor function. Obviously, if $n_1 \geq 7$, then $2 \leq n_1 \leq 6$. It is also because the least prime factors power of n_1 is bigger than 1, we have $n_1 = 4$, and $n = n_1 p = 4p$, $p > 3$. Now, from

$$4p = \sum_{d|4p} \overline{S}(d) = \overline{S}(1) + \overline{S}(2) + \overline{S}(4) + \overline{S}(p) + \overline{S}(2p) + \overline{S}(4p) = 1 + 2 + 4 + 3p,$$

we immediately obtain $p = 7$ and $n = 28$.

(B) $\overline{S}(n) = \alpha p$ and $\alpha > 1$, now let $n = n_1 p^\alpha, (n_1, p) = 1$, if n satisfied (2), we have

$$n = p^\alpha n_1 = \sum_{i=0}^{\alpha} \sum_{d|n_1} \overline{S}(p^i d).$$

If $1 < n_1 < 8$, we consider equation (2) as follows:

(a) If $n_1 = 2$. That is, $n = 2p^\alpha (p > 2)$, from the discussion of (iii), we know that $n = 2p^\alpha$ isn't the solution of (2).

(b) If $n_1 = 3$. That is, $n = 3p^\alpha$, since $(n_1, p) = 1$, we have $p \neq 3$.

If $p = 2, n = 3 \cdot 2^\alpha$ satisfied (2). That is

$$\sum_{d|3\cdot 2^\alpha} \overline{S}(d) = \sum_{d|2^\alpha} \overline{S}(d) + \sum_{d|2^\alpha} \overline{S}(3d) = 2 \sum_{d|2^\alpha} \overline{S}(d) + 3 = 3 \cdot 2^\alpha.$$

In the above equation, $2 \sum\limits_{d|2^\alpha} \overline{S}(d) + 3$ is an odd number, but $3 \cdot 2^\alpha$ is an even number, so $n = 3 \cdot 2^\alpha$ is not the solution of (2).

If $p > 3$. That is, $n = 3 \cdot p^\alpha$ satisfied (2), then the least prime factor powers of n is 1, from (iii), we know that $n = 3 \cdot p^\alpha$ is not the solution of (2).

(c) If $n_1 = 4$, $n = 4 \cdot p^\alpha (p \geq 3)$, we have

$$\sum_{d|4\cdot p^\alpha} \overline{S}(d) = \sum_{d|p^\alpha} \overline{S}(d) + \sum_{d|p^\alpha} \overline{S}(2d) + \sum_{d|p^\alpha} \overline{S}(4d).$$

If $p=3$. That is, $n=4\cdot 3^{\alpha}$ satisfied equation (2), then

$$\sum_{d|4\cdot 3^{\alpha}}\overline{S}(d)=\sum_{d|3^{\alpha}}\overline{S}(d)+\sum_{d|3^{\alpha}}\overline{S}(2d)+\sum_{d|3^{\alpha}}\overline{S}(4d)=3\sum_{\substack{d|3^{\alpha}\\ d>1}}\overline{S}(d)+12=4\cdot 3^{\alpha}.$$

Since $3^2 \mid 3\sum\limits_{\substack{d|3^{\alpha}\\ d>1}}\overline{S}(d)$, and $3^2\mid 4\cdot 3^{\alpha}$, then $3^2\mid 12$, this is impossible.

If $p>3$. That is, $n=4\cdot p^{\alpha}$, then

$$\sum_{d|4\cdot p^{\alpha}}\overline{S}(d)=\sum_{d|p^{\alpha}}\overline{S}(d)+\sum_{d|p^{\alpha}}\overline{S}(2d)+\sum_{d|p^{\alpha}}\overline{S}(4d)=3\sum_{d|p^{\alpha}}\overline{S}(d)+8=\frac{3}{2}\alpha(\alpha+1)p+11=4\cdot p^{\alpha},$$

or $4\cdot 3^{\alpha}-\frac{3}{2}\alpha(\alpha+1)p+11=0$. Now we fix α, and let $f(x)=4\cdot x^{\alpha}-\frac{1}{2}\alpha(\alpha+1)x+11$, if $x\geq 3$, $f(x)$ is a increased function. That is,

$$f(x)\geq f(3)=4\cdot 3^{\alpha}-\frac{3}{2}\alpha(\alpha+1)+11=g(\alpha).$$

So when $x\geq 3$, $f(x)=0$ has no solutions, from which we get if $p>3$, then equation (2) has no solutions.

(d) If $n_1=5$, we have $n=5\cdot p^{\alpha}(p\neq 5)$.

If $p>5$, then from (iii), we know that $n=5\cdot p^{\alpha}$ is not a solution of equation (2).

If $p=2$, since

$$\sum_{d|5\cdot 2^{\alpha}}\overline{S}(d)=\sum_{d|2^{\alpha}}\overline{S}(d)+\sum_{d|2^{\alpha}}\overline{S}(5d)=2\sum_{\substack{d|2^{\alpha}\\ d>1}}\overline{S}(d)+10=5\cdot 2^{\alpha},$$

where $2^2\mid 2\sum\limits_{\substack{d|2^{\alpha}\\ d>1}}\overline{S}(d)$, and $2^2\mid 5\cdot 2^{\alpha}$, so we have $2^2\mid 10$, this is impossible. Hence $n=5\cdot 2^{\alpha}$ unsatisfied the equation (2).

If $p=3$, since

$$\sum_{d|5\cdot 3^{\alpha}}\overline{S}(d)=\sum_{d|3^{\alpha}}\overline{S}(d)+\sum_{d|3^{\alpha}}\overline{S}(5d)=2\sum_{d|3^{\alpha}}\overline{S}(d)+6,$$

where $2\sum\limits_{d|3^{\alpha}}\overline{S}(d)+6$ is even, and $5\cdot 3^{\alpha}$ is odd, so $n=5\cdot 3^{\alpha}$ unsatisfied the equation (2).

(e) When $n_1=6$, $n=2\cdot 3\cdot p^{\alpha}$, from the discussion of (3), n unsatisfied (2).

(f) When $n_1=7$, we have $n=7\cdot p^{\alpha}(p\neq 7)$.

If $p>7$, then from (iii), $n=7\cdot p^{\alpha}$ isn't the solution of (2).

If $p=2$, we must have $\alpha\geq 4$. Since

$$\sum_{d|7\cdot 2^{\alpha}}\overline{S}(d)=\sum_{d|2^{\alpha}}\overline{S}(d)+\sum_{d|2^{\alpha}}\overline{S}(7d)=2\sum_{d|2^{\alpha}}\overline{S}(d)+15,$$

where $2\sum\limits_{d|2^{\alpha}}\overline{S}(d)+15$ is odd, but $n=7\cdot 2^{\alpha}$ is even. So $n=7\cdot 2^{\alpha}$ unsatisfied (2).

If $p=3$, since

$$\sum_{d|7\cdot 3^{\alpha}}\overline{S}(d)=\sum_{d|3^{\alpha}}\overline{S}(d)+\sum_{d|3^{\alpha}}\overline{S}(7d)=2\sum_{\substack{d|3^{\alpha}\\ d>1}}\overline{S}(d)+13,$$

in above equation, $3 \mid 2 \sum\limits_{\substack{d \mid 3^\alpha \\ d>1}} \overline{S}(d)$, and $3 \mid 7 \cdot 3^\alpha$. If it satisfied (2), we must obtain $3 \dagger 13$, a contradiction! So $n = 7 \cdot 3^\alpha$ is not a solution of (2) either.

If $p = 5$, since

$$\sum_{d \mid 7 \cdot 5^\alpha} \overline{S}(d) = \sum_{d \mid 5^\alpha} \overline{S}(d) + \sum_{d \mid 5^\alpha} \overline{S}(7d) = 2 \sum_{d \mid 5^\alpha} \overline{S}(d) + 8,$$

in the above equation, $2 \sum\limits_{d \mid 5^\alpha} \overline{S}(d) + 8$ is even, $7 \cdot 5^\alpha$ is odd. So $n = 7 \cdot 5^\alpha$ is not a solution of (2) either.

(g) When $n_1 \geq 8$, we have $n = n_1 \cdot p^\alpha$ and $p^\alpha > \frac{\alpha(\alpha+1)}{2} p$, then

$$\sum_{d \mid n_1 \cdot p^\alpha} \overline{S}(d) < \overline{S}(p^\alpha) d(n_1 p^\alpha) = \alpha(\alpha+1) p d(n_1) \leq \frac{\alpha(\alpha+1)}{2} p n_1 < p^\alpha n_1 = n,$$

then if $n_1 \geq 8$, $n = n_1 p^\alpha$ is not a solution of (2) either.

In a word, equation (2) only has two solutions $n = 1$ and $n = 28$.

This completes the proof of the theorem.

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