On the Irrationality of Certain Constants Related to the Smarandache Function

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1. Let $S(n)$ be the Smarandache function. Recently I. Cojocaru and S. Cojocaru [2] have proved the irrationality of $\sum_{n=1}^{n} \frac{S(n)}{n!}$.

The author of this note [5] showed that this is a consequence of an old irrationality criteria (which will be used here once again), and proved a result implying the irrationality of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{S(n)}{n!}$.

E. Burton [1] has studied series of type $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$, which has a value $\in \left(e - \frac{5}{2}, \frac{1}{2}\right)$. He showed that the series $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!}$ is convergent for all $r \in \mathbf{N}$. I. Cojocaru and S. Cojocaru [3] have introduced the "third constant of Smarandache" namely $\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\ldots S(n)}$, which has a value between $\frac{71}{100}$ and $\frac{97}{100}$. Our aim in the following is to prove that the constants introduced by Burton and Cojocaru-Cojocaru are all irrational.

2. The first result is in fact a refinement of an old irraionality criteria (see [4] p.5):

Theorem 1. *Let (x_n) be a sequence of nonnegative integers having the properties:*

(1) there exists $n_0 \in \mathbf{N}^*$ such that $x_n \le n$ for all $n \ge n_0$;

(2) $x_n < n - 1$ for infinitely many n;

(3) $x_m > 0$ for an infinity of m.

Then the series $\sum_{n=1}^{\infty} \frac{x_n}{n!}$ *is irrational.*

Let now $x_n = S(n-1)$. Then

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!} = \sum_{n=3}^{\infty} \frac{x_n}{n!}.$$

Here $S(n-1) \leq n-1 < n$ for all $n \geq 2$; $S(m-1) < m-2$ for $m > 3$ composite, since by $S(m-1) < \frac{2}{3}(m-1) < m-2$ for $m > 4$ this holds true. (For the inequality $S(k) < \frac{2}{3}k$ for $k > 3$ composite, see [6]). Finally, $S(m-1) > 0$ for all $m \geq 1$. This proves the irrationality of $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$.

Analogously, write

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} = \sum_{m=r+2}^{\infty} \frac{S(m-r)}{m!}.$$

Put $x_m = S(m-r)$. Here $S(m-r) \leq m-r < m$, $S(m-r) \leq m-r < m-1$ for $r \geq 2$, and $S(m-r) > 0$ for $m \geq r+2$. Thus, the above series is irrational for $r \geq 2$, too.

3. The third constant of Smarandache will be studied with the following irrationality criterion (see [4], p.8):

Theorem 2. *Let* $(a_n), (b_n)$ *be two sequences of nonnegative integers satisfying the following conditions:*

(1) $a_n > 0$ *for an infinity of* n*;*

(2) $b_n \geq 2$, $0 \leq a_n \leq b_n - 1$ *for all* $n \geq 1$*;*

(3) there exists an increasing sequence (i_n) *of positive integers such that*

$$\lim_{n\to\infty} b_{i_n} = +\infty, \quad \lim_{n\to\infty} a_{i_n}/b_{i_n} = 0.$$

Then the series $\sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \ldots b_n}$ *is irrational.*

Corollary. *For* $b_n \geq 2$, *(*b_n *positive integers),* (b_n) *unbounded the series* $\sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \ldots b_n}$ *is irrational.*

Proof. Let $a_n \equiv 1$. Since $\limsup_{n\to\infty} b_n = +\infty$, there exists a sequence (i_n) such that $b_{i_n} \to \infty$. Then $\frac{1}{b_{i_n}} \to 0$, and the three conditions of Theorem 2 are verified.

By selecting $b_n \equiv S(n)$, we have $b_p = S(p) = p \to \infty$ for p a prime, so by the above Corollary, the series $\sum_{n=1}^{\infty} \frac{1}{S(1)S(2)\dots S(n)}$ is irrational.

References

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[2] I. Cojocaru and S. Cojocaru, *The second constant of Smarandache*, Smarandache Notions J. **7**(1996), no.1-2-3, 119-120.

[3] I. Cojocaru and S. Cojocaru, *The third and fourth constants of Smarandache*, Smarandache Notions J. **7**(1996), no.1-2-3, 121-126.

[4] J. Sándor, *Irrational Numbers* (Romanian), Univ. Timişoara, Caiete Metodico-Stiinţifice No.44, 1987, pp. 1-18.

[5] J. Sándor, *On the irrationality of certain alternative Smarandache series*, Smarandache Notion J. **8**(1997), no.1-2-3, 1997, 143-144.

[6] T. Yau, *A problem of maximum*, Smarandache Function J., vol. **4-5**(1994), no.1, p.45.