

All Solutions of the Equation $S(n) + d(n) = n$

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The number of divisors function $d(n)$, is a classic function of number theory, having been defined centuries ago. In contrast, the Smarandache function $S(n)$, was defined only a few decades ago. The purpose of this paper is to find all solutions to a simple equation involving both functions.

Theorem: The only solutions to the equation

$$S(n) + d(n) = n, \quad n > 0$$

are 1, 8 and 9.

Proof: Since $S(1) = 0$ and $d(1) = 1$ we have verified the special case of $n = 1$.

Furthermore, with $S(p) = p$ for p a prime, it follows that any solution must be composite.

The following results are well-known.

- a) $d(p_1^{a_1} \dots p_k^{a_k}) = (a_1 + 1) \dots (a_k + 1)$
- b) $S(p^k) \leq kp$
- c) $S(p_1^{a_1} \dots p_k^{a_k}) = \max \{ S(p_1^{a_1}) \dots S(p_k^{a_k}) \}$

Examining the first few powers of 2.

$$\begin{aligned} S(2^2) &= 4, \quad d(2^2) = 3 \\ S(2^3) &= 4 \text{ and } d(2^3) = 4 \text{ which is a solution.} \\ S(2^4) &= 6, \quad d(2^4) = 5 \end{aligned}$$

and in general

$$S(2^k) \leq 2k \quad \text{and} \quad d(2^k) = k + 1.$$

It is an easy matter to verify that

$$2k + k + 1 = 3k + 1 < 2^k$$

for $k > 4$.

Examining the first few powers of 3

$$S(3^2) = 6 \text{ and } d(3^2) = 3, \text{ which is a solution.}$$
$$S(3^3) = 9, d(3^3) = 4$$

and in general, $S(3^k) \leq 3k$ and $d(3^k) = k + 1$.

It is again an easy matter to verify that

$$3k + k + 1 < 3^k$$

for $k > 3$.

Consider $n = p^k$ where $p > 3$ is prime and $k > 1$. The expression becomes

$$S(p^k) + d(p^k) \leq kp + k + 1 = k(p+1) + 1.$$

Once again, it is easy to verify that this is less than p^k for $p \geq 5$.

Now, assume that $n = p_1^{a_1} \dots p_k^{a_k}$, $k > 1$ is the unique prime factorization of n .

Case 1: $n = p_1 p_2$, where $p_2 > p_1$. Then $S(n) = p_2$ and $d(n) = 2 * 2 = 4$. Forming the sum,

$$p_2 + 4$$

we then examine the subcases.

Subcase 1: $p_1 = 2$. The first few cases are

$$n = 2 * 3, S(n) + d(n) = 7$$
$$n = 2 * 5, S(n) + d(n) = 9$$
$$n = 2 * 7, S(n) + d(n) = 11$$
$$n = 2 * 11, S(n) + d(n) = 15$$

and it is easy to verify that $S(n) + d(n) < n$, for p_2 a prime greater than 11.

Subcase 2: $p_1 = 3$. The first few cases are

$$n = 3 * 5, S(n) + d(n) = 5 + 4$$
$$n = 3 * 7, S(n) + d(n) = 7 + 4$$
$$n = 3 * 11, S(n) + d(n) = 11 + 4$$

and it is easy to verify that $S(n) + d(n) < n$ for p_2 a prime greater than 11.

Subcase 3: It is easy to verify that

$$p_2 + 4 < p_1 p_2$$

for $p_1 \geq 5, p_2 > p_1$.

Therefore, there are no solutions for $n = p_1 p_2, p_1 < p_2$.

Case 2: $n = p_1 p_2^{a_2}$, where $a_2 > 1$ and $p_1 < p_2$. Then $S(n) \leq a_2 p_2$ and $d(n) = 2(a_2 + 1)$.

$$S(n) + d(n) \leq a_2 p_2 + 2(a_2 + 1) = a_2 p_2 + 2a_2 + 2$$

We now induct on a_2 to prove the general inequality

$$a_2 p_2 + 2a_2 + 2 < p_1 p_2^{a_2}$$

Basis step: $a_2 = 2$. The formula becomes

$$2p_2 + 4 + 2 = 2p_2 + 6 \text{ on the left and}$$

$p_1 p_2 p_2$ on the right. Since $p_2 \geq 3, 2 + \frac{6}{p_2} \leq 4$ and $p_1 p_2 \geq 6$. Therefore,

$$2 + \frac{6}{p_2} < p_1 p_2$$

and if we multiply everything by p_2 , we have

$$2p_2 + 6 < p_1 p_2 p_2.$$

Inductive step: Assume that the inequality is true for $k \geq 2$

$$k p_2 + 2k + 2 < p_1 p_2^k.$$

and examine the case where the exponent is $k + 1$.

$$(k + 1)p_2 + 2(k + 1) + 2 = k p_2 + p_2 + 2k + 2 + 2 = (k p_2 + 2k + 2) + p_2 + 2$$

$$< p_1 p_2^k + p_2 + 2 \quad \text{by the inductive hypothesis.}$$

Since $p_1 p_2^k$ when $k \geq 2$ is greater than $p_2 + 2$ it follows that

$$p_1 p_2^k + p_2 + 2 < p_1 p_2^{k+1}.$$

Therefore, $S(n) + d(n) < n$, where $n = p_1 p_2^k, k \geq 2$.

Case 3: $n = p_1^{a_1} p_2$, where $a_1 > 1$.

We have two subcases for the value of $S(n)$, depending on the circumstances

Subcase 1: $S(n) \leq a_1 p_1$.

Subcase 2: $S(n) = p_2$.

In all cases, $d(n) = 2(a_1 + 1)$.

Subcase 1: $S(n) + d(n) \leq a_1 p_1 + 2(a_1 + 1) = a_1 p_1 + 2a_1 + 2$.

Using an induction argument very similar to that applied in case 2, it is easy to prove that the inequality

$$a_1 p_1 + 2a_1 + 2 < p_1^{a_1} p_2.$$

is true for all $a_1 \geq 2$.

Subcase 2: $S(n) + d(n) = p_2 + 2(a_1 + 1) = p_2 + 2a_1 + 2$

It is again a simple matter to verify that the inequality

$$p_2 + 2a_1 + 2 < p_1^{a_1} p_2$$

is true for all $a_1 \geq 2$.

Case 4: $n = p_1^{a_1} p_2^{a_2}$, where $p_1 < p_2$ and $a_1, a_2 \geq 2$.

$d(n) = (a_1 + 1)(a_2 + 1)$

Subcase 1: $S(n) \leq a_1 p_1$

$$S(n) + d(n) \leq a_1 p_1 + (a_1 + 1)(a_2 + 1) < p_1^{a_1} + p_1^{a_1}(a_2 + 1) = p_1^{a_1}(a_2 + 2) < p_1^{a_1} p_2^{a_2}$$

Subcase 2: $S(n) \leq a_2 p_2$

$$S(n) + d(n) \leq a_2 p_2 + (a_1 + 1)(a_2 + 1) < p_2^{a_2} + p_2^{a_2}(a_1 + 1) = p_2^{a_2}(a_1 + 2) < p_1^{a_1} p_2^{a_2}$$

Case 5: $n = p_1^{a_1} \dots p_k^{a_k}$, where $k \geq 2$.

The proof is by induction on k .

Basis step: Completed in the first four cases.

Inductive step: Assume that for $n_1 = p_1^{a_1} \dots p_k^{a_k}$, $k \geq 2$

$$a_i p_i + (a_1 + 1) \dots (a_k + 1) < n_1$$

where $S(n_1) \leq a_i p_i$. Which means that

$$S(n_1) + d(n_1) < n_1.$$

Consider $n_2 = p_1^{a_1} \dots p_k^{a_k} p_{k+1}^{a_{k+1}}$.

Subcase 1: $S(n_2) = S(n_1)$. Since $p_{k+1} \geq 5$, it follows that $(a_{k+1} + 1) < p_{k+1}^{a_{k+1}}$ and we can this in combination with the inductive hypothesis to conclude

$$a_i p_i + (a_1 + 1) \dots (a_k + 1)(a_{k+1} + 1) < n_1 p_{k+1}^{a_{k+1}},$$

which implies that $S(n_2) + d(n_2) < n_2$.

Subcase 2: $S(n_2) > S(n_1)$, which implies that $S(n_2) \leq a_{k+1} p_{k+1}$. Starting with the inductive hypotheses

$$a_i p_i + (a_1 + 1) \dots (a_k + 1) < p_1^{a_1} \dots p_k^{a_k}$$

and multiply both sides by $a_{k+1} p_{k+1}$ to obtain the inequality

$$a_i p_i a_{k+1} p_{k+1} + a_{k+1} p_{k+1} (a_1 + 1) \dots (a_k + 1) < p_1^{a_1} \dots p_k^{a_k} a_{k+1} p_{k+1}$$

Since $p_{k+1} \geq 5$, it follows that

$$p_1^{a_1} \dots p_k^{a_k} a_{k+1} p_{k+1} \leq p_1^{a_1} \dots p_k^{a_k} p_{k+1}^{a_{k+1}}$$

and with $a_{k+1} p_{k+1} > (a_{k+1} + 1)$, we have

$$\begin{aligned} a_{k+1} p_{k+1} + (a_1 + 1) \dots (a_k + 1)(a_{k+1} + 1) < \\ a_i p_i a_{k+1} p_{k+1} + a_{k+1} p_{k+1} (a_1 + 1) \dots (a_k + 1). \end{aligned}$$

Combining the inequalities, we have

$$a_{k+1} p_{k+1} + (a_1 + 1) \dots (a_k + 1)(a_{k+1} + 1) < p_1^{a_1} \dots p_k^{a_k} p_{k+1}^{a_{k+1}}$$

which implies

$$S(n_2) + d(n_2) < n.$$

Therefore, the only solutions to the equation

$$S(n) + d(n) = n$$

are 1, 8 and 9.