THE ANALYTICAL FORMULAE YIELDING SOME SMARANDACHE NUMBERS AND APPLICATIONS IN MAGIC SQUARES THEORY

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In this paper we study the properties of some six numerical Smarandache sequences. As result we present a set of analytical formulae for the computation of numbers in these Smarandache series and for constructing Magic squares 3×3 in size from k-truncated Smarandache numbers. The examples of Magic squares 3×3 in size of six Smarandache sequences are also adduced.

1 Introduction

In this paper some properties of six different Smarandache sequences of the 1st kind[1] are investigated. In particular, as we stated, the terms of these six sequences may be computed by means of one general recurrent expression

$$a_{\varphi(n)} = \sigma(a_n 10^{\psi(a_n)} + a_n + 1), \tag{1}$$

where a_n — n-th number of Smarandache sequence; $\varphi(n)$ and $\psi(a_n)$ — some functions; σ — an operator. For each of six Smarandache sequences, determined by (1), we adduce (see Sect. 2 and 3)

a) several first numbers of the sequence;

b) the concrete form of the analytical formula (1);

c) the analytical formula for the calculation of n-th number in the sequence;

d) a set of analytical formulae for constructing Magic squares 3×3 in size from k-truncated Smarandache numbers;

e) a few of concrete examples of Magic squares 3×3 in size from k-truncated Smarandache numbers.

2 Analytical formulae yielding Smarandache sequences

1. Smarandache numbers of S_1-series. If $\varphi(n) = n+1$, $\sigma = 1$ and $\psi(a_n) = [\lg(n+1)]+1$ then of (1) the following series of the numbers, denoted as S_1-*series*, is generated

$$1,\ 12,\ 123,\ 1234,\ 12345,\ 123456, \ldots \tag{2}$$

The each number of

$$\chi_k = -1 + \sum_{j=0}^{[\lg(k+0,5)]} (k+1-10^j), \tag{3}$$

corresponds to each number a_k of series (2), where the notation "$[\lg(y)]$" means integer part from decimal logarithm of y. By (3) it is easy to construct the analytical formula for the calculation of n-th number in the S_1-series:

$$a_n = 10^{\chi_n} \sum_{i=1}^{n} (i / 10^{\chi_i}). \tag{4}$$

By expressions

$$\Lambda^0 a_n = 1234\ldots(n-1)n;\ \Lambda^{-1} a_n = 234\ldots(n-1)n;\ \Lambda^{-2} a_n = 34\ldots(n-1)n;\ \ldots \tag{5}$$

we introduce the operator Λ^{-k} {the operator of *k-truncating* the number $a_n = 1234\ldots(n-1)n$}. Since

$$\Lambda^0 a_1 = 1,\ \Lambda^{-1} a_2 = 2,\ \Lambda^{-2} a_3 = 3,\ \ldots,\ \Lambda^{-n+1} a_n = n,\ \ldots \tag{6}$$

it is evident that by the operator Λ^{-k} from the numbers of S_1-series one may produce the series of the natural numbers. And, vice versa, if the operator Λ^{+k} {the operator of *k-extending* the number n} is introduced:

$$\Lambda^0 n = n;\quad \Lambda^{+1} n = (n-1)n;\quad \Lambda^{+2} n = (n-2)(n-1)n;\ \ldots \tag{7}$$

then from the series of the natural numbers one may obtain the numbers of S_1-series:

$$\Lambda^{0}\, 1 = a_1,\quad \Lambda^{+1}\, 2 = a_2,\quad \Lambda^{+2}\, 3 = a_3,\ \ldots\ ,\quad \Lambda^{n-1}\, n = a_n,\ \ldots \tag{8}$$

It is evident that

a) the operators Λ^{+k} and Λ^{-k} are connected with each to other. Therefore one may simplify their arbitrary combinations by the mathematical rule of the action with the power expressions {for instance, $\Lambda^{+2}\Lambda^{-7}\Lambda^{+3}= \Lambda^{+2-7+3}=\Lambda^{-2}$}.

b) apart from operators of k-truncating and k-extending of numbers *from the left* {see (5) and (7)} one may introduce operators of k-truncating and k-extending of numbers *from the right* {for instance, $(\Lambda^{-2}\, 12345) = 345$, but $(12345\, \Lambda^{-2}) = 123$};

c) by means of operators of k-truncating and k-extending of numbers from the right one may represent the different relations existing between the numbers of S_1-series {for instance, $a_n = (a_{n-1}\, \Lambda^{+1}) = (a_{n+1}\, \Lambda^{-1})$ and so on}.

2. Smarandache numbers of S_2-series. If $\varphi(n)=n+1$; $\sigma = \gamma$ — the operator of mirror-symmetric extending the number $a_{[(n+1)/2]}$ of S_1-series from the right with 1-truncating the reflected number from the left, if n is the odd number, and without truncating the reflected number, if n is the even number; $\psi(a_n) = [\lg([(n+1)/2] + 1)] + 1$, then of (1) the following series of the numbers, denoted as *S_2-series*, is generated

$$1,\ 11,\ 121,\ 1221,\ 12321,\ 123321,\ 1234321, \ldots \tag{9}$$

The analytical formula for the calculation of n-th number in the S_2-series has the form

$$a_n = \sum_{i=1}^{[n/2]} i\, 10^{\chi_i - [\lg i]} + \sum_{i=1}^{[(n+1)/2]} i\, 10^{d}\ , \tag{10}$$

where $d = 1 + \chi_{[(n+1)/2]} + \chi_{[n/2]} - \chi_i$.

3. Smarandache numbers of S_3-series. If $\varphi(n) = n+1$; $\sigma = \gamma$ — the operator of mirror-symmetric extending the number a_n of S_1-series from the left with 1-truncating the reflected number from the right; $\psi(a_n) = [\lg(n + 1)] + 1$, then of (1) the following series of the numbers, denoted as *S_3-series*, is generated

$$1,\ 212,\ 32123,\ 4321234,\ 543212345,\ 65432123456, \ldots \tag{11}$$

The analytical formula for the calculation of n-th number in the S_3-series has the form

$$a_n = 10^{\chi_n} \{ \sum_{i=2}^{n} (i\, 10^{\chi_i}) / 10^{[\lg i]} + \sum_{i=1}^{n} i / 10^{\chi_i} \} . \tag{12}$$

4. Smarandache numbers of S_4-series. The series of the numbers

$$1,\ 23,\ 456,\ 7891,\ 23456,\ 789123,\ 4567891, \ldots \tag{13}$$

we denote as *S_4-series.* It is evident that the series of the numbers (13) is obtained from the infinite circular chain of the numbers

$$(123456789)(123456789) \ldots (123456789) \ldots \tag{14}$$

by means of the proper truncation from the left and the right. The analytical formula for the calculation of n-th number in the S_4-series has the form

$$a_n = 10^n \sum_{i=0}^{n-1} \{ 1 + d - 9\,[d/9] \} / 10^{i+1}, \quad d = i + n\,(n-1)/2. \tag{15}$$

5. Smarandache numbers of S_5-series. The series of the numbers

$$1,\ 12,\ 21,\ 123,\ 231,\ 312,\ 1234,\ 2341,\ 3412,\ 4123,\ 12345, \ldots \tag{16}$$

we denote as *S_5-series.* By (3) it is easy to construct the analytical formula for the calculation of n-th number in the S_5-series:

$$a_n = \sum_{i=1}^{z} (i\, 10^d), \quad z = [(\sqrt{8n-7} - 1)/2], \tag{17}$$
$$d = \chi_t - \chi_i - (\chi_z + 1)\, [(\chi_t - \chi_i)/(\chi_z + 1)], \quad t = -1 + n - z(z-1)/2.$$

6. Smarandache numbers of S_6-series. The series of the numbers

$$12,\ 1342,\ 135642,\ 13578642,\ 13579108642, \ldots \tag{18}$$

we denote as S_6*-series*. The analytical formula for the calculation of n-th number in the S_6-series has the form

$$a_n = \{10^{1+2[\chi_{2n}/2]} \sum_{i=1}^{n} (2i-1) / 10^{[\chi_{2i-1}/2]} + \sum_{i=1}^{n} 2i\, 10^{[\chi_{2i}/2]}\} / 10^{[\lg 2n]} . \tag{19}$$

3 Magic squares 3×3 in size from k-truncated Smarandache numbers

1. Magic squares 3×3 in size from k-truncated numbers of S_1*-series*. By analysing numbers a_n of S_1-series one can conclude that it is impossible to construct an arithmetical progression from any three numbers of S_1-series. Consequently[2], none Magic square 3×3 in size can be constructed from these numbers. However, one may truncate number a_n of S_1-series from the left or/and the right by means of the operator Λ^{-k}(5). Therefore there is a possibility to construct the Magic squares 3×3 in size from truncated numbers of S_1-series. In particular, the analytical formula for constructing such Magic squares is adduced in the Fig. 1(1). If in the formula 1(1) the parameters n, r, p and q take, for instance, the following values:

a) $n = 7$, $r = 14$, $p = 1$ and $q = 3$, then it generates the Magic square 3×3 shown in the Fig. 1(2);

b) $n = 4$, $r = 0$, $p = 1$ and $q = 3$, then the numerical square 3×3, shown in the Fig. 1(3), is yielded — the square 1(3) is not Magic, but it can be easy transformed to one by means of revising three numbers marked out by the dark background {the revised square see in Fig. 1(3′)};

c) $n = 4$, $r = 7$, $p = 1$ and $q = 3$, then the numerical square 3×3, shown in the Fig. 1(5), is yielded — the square 1(5) also is not Magic, but it can be easy transformed to one by means of revising just one number marked out by the dark background {the revised square see in Fig. 1(5′)}.

By analysing the squares, shown in the Fig. 1(3) and 1(5), it can be easy understood that the analytical formula 1(1) does not hold true only in such cases when natural numbers, being components of numbers $\Lambda^{-k} a_n$, have different amount of digits. To obtain the Magic square in this case, one is to correct the defects of the square generated by formula 1(1) {as it made, for instance, in Fig. 1(3′) and 1(5′) for squares 1(3) and 1(5)}, or to change the values of parameters n, r, p and/or q correspondingly.

$\Lambda^{-r-p-2q}\, a_{n+r+p+2q}$	$\Lambda^{-r}\, a_{n+r}$	$\Lambda^{-r-2p-q}\, a_{n+r+2p+q}$
$\Lambda^{-r-2p}\, a_{n+r+2p}$	$\Lambda^{-r-p-q}\, a_{n+r+p+q}$	$\Lambda^{-r-2q}\, a_{n+r+2q}$
$\Lambda^{-r-q}\, a_{n+r+q}$	$\Lambda^{-r-2p-2q}\, a_{n+r+2p+2q}$	$\Lambda^{-r-p}\, a_{n+r+p}$

(1)

22232425262728	15161718192021	20212223242526
17181920212223	19202122232425	21222324252627
18192021222324	23242526272829	16171819202122

(2)

891011	1234	6789
3456	5678	78910
4567	9101112	2345

(3)

9011	1234	6789
3456	5678	8900
4567	10122	2345

(3′)

38	10	30
18	26	34
22	42	14

(4)

15161718	891011	13141516
10111213	12131415	14151617
11121314	16171819	9101112

(5)

15161718	891011	13141516
10111213	12131415	14151617
11121314	23371819	9101112

(5′)

$(r+p+2q)\,n + n(n+1)/2$	$r\,n + n(n+1)/2$	$(r+2p+q)\,n + n(n+1)/2$
$(r+2p)\,n + n(n+1)/2$	$(r+p+q)\,n + n(n+1)/2$	$(r+2q)\,n + n(n+1)/2$
$(r+q)\,n + n(n+1)/2$	$(r+2p+2q)\,n + n(n+1)/2$	$(r+p)\,n + n(n+1)/2$

(6)

Fig. 1. Constructing Magic squares 3×3 from k-truncated numbers of S_1-series.

It should be noted that the proper replacement of numbers $\Lambda^{-k}a_n$ in squares 1(2), 1(3) and 1(5) by the sum of digits of natural numbers, being components of $\Lambda^{-k}a_n$, gives three different Magic squares 3×3. For instance, the Magic square, obtained by such way from square 1(3), is depicted in Fig. 1(4). The explanation of this curious fact can be found in Fig. 1(6), presenting the analytical formula of Magic square 3×3, which is obtained directly from the formula 1(1) by means of the mentioned way.

2. Magic squares 3×3 in size from k-truncated numbers of S_2-series. To apply the methods, elaborated in point 1, for constructing Magic squares 3×3 from numbers of S_2-series {see (9)}, we divide a set of S_2-series numbers into two different subsequences:

1) $a_1=1,\ a_2=121,\ a_3=12321,\ a_4=1234321, \ldots$

2) $b_1=11,\ b_2=1221,\ b_3=123321,\ b_4=12344321, \ldots$

By adding to the all elements of the analytical formula 1(1) from the right the operator Λ^{-k}, having the same form as one located from the left, we obtain the new formula of the Magic square 3×3. This formula allows easy to construct examples of Magic squares 3×3 both from numbers of the first subsequence {see Fig. 2(1)} and from numbers of the second subsequence {see Fig. 2(2)}.

171819191817	101112121110	151617171615
121314141312	141516161514	161718181716
131415151413	181920201918	111213131211

(1)

17181920191817	10111213121110	15161718171615
12131415141312	14151617161514	16171819181716
13141516151413	18192021201918	11121314131211

(2)

Fig. 2. Constructing Magic squares 3×3 from k-truncated numbers of S_2-series.

3. Magic squares 3x3 in size from k-truncated numbers of S_3-series. By comparing numbers of S_3-series {see (11)} and S_2-series {see point 2} with each to other one can conclude that numbers of S_3-series resemble numbers of the first subsequence of S_2-series and distinguish from them on the order of the natural numbers movement. The example of the Magic square 3x3 from numbers of S_3-series is presented in Fig. 3. This square is constructed by means of methods described in point 1 and 2. Thus, in spite of the mentioned difference between numbers of S_3-series and S_2-series, the methods, discussed above, can be applied for solving problems on constructing Magic square 3x3 from numbers of S_3-series.

201918181920	131211111213	181716161718
151413131415	171615151617	191817171819
161514141516	212019192021	141312121314

Fig. 3. Constructing Magic squares 3x3 from k-truncated numbers of S_3-series.

4. Magic squares 3x3 in size from k-truncated numbers of S_4-series. In contrast to considered Smarandache sequences the digit 0 is absent in numbers of S_4- series. Besides, the order of the movement for digits 1, 2, ..., 9 can not be changed and after digit 9 can be the only digit 1. These peculiarities of numbers of S_4- series make too difficult the solving problems on constructing Magic square 3x3. It is evident that by using Λ^{-k}-operator one can easy construct *classical* square 4(1) {the Magic square of natural numbers from 1 to 9}. Since by means of Λ^{-k}-operator such square can be constructed from numbers of any Smarandache sequence {for instance, see (6)}, the example of the square 4(1) is banal. The example of the Magic square 3x3, presented in Fig. 4(2, 3), is less trivial.

8	1	6
3	5	7
4	9	2

(1)

$a_4\Lambda^{-2}$	a_1	$\Lambda^{-1}a_3$
a_2	$a_3\Lambda^{-1}$	$\Lambda^{-2}a_7\Lambda^{-3}$
$\Lambda^{-1}a_5\Lambda^{-2}$	$\Lambda^{-1}a_4\Lambda^{-1}$	$\Lambda^{-3}a_6\Lambda^{-1}$

(2)

78	1	56
23	45	67
34	89	12

(3)

Fig. 4. Constructing Magic squares 3x3 from k-truncated numbers of S_4-series.

5. Magic squares 3x3 in size from k-truncated numbers of S_5-series. As compared with another Smarandache sequences of the 1st kind the numbers of S_5-series {see (16)} have the following peculiarity: the circular permutation of natural numbers is allowed in them. The analytical formula of Magic square 3x3, presented in Fig. 5(1), is just constructed with taking into account the pointed peculiarity of discussed numbers. Examples of the Magic square 3x3, obtained from formula 5(1) at $n = 2$, 3 and 4, are depicted in Fig. 5(2, 3, 4) correspondingly. By analysing these squares it is easy to find more simple form of the analytical formula 5(1) {see Fig. 5(5), where a_{n-1} is the $(n-1)$th number of S_1-series, M is general amount of digits in the number a_{n-1}}.

$\Lambda^{-7} a_{n(n+15)/2+21}$	$a_{n(n+1)/2}$	$\Lambda^{-5} a_{n(n+11)/2+10}$
$\Lambda^{-2} a_{n(n+5)/2+1}$	$\Lambda^{-4} a_{n(n+9)/2+6}$	$\Lambda^{-6} a_{n(n+13)/2+15}$
$\Lambda^{-3} a_{n(n+7)/2+3}$	$\Lambda^{-8} a_{n(n+17)/2+28}$	$\Lambda^{-1} a_{n(n+3)/2}$

(1)

91	21	71
41	61	81
51	101	31

(2)

1012	312	812
512	712	912
612	1112	412

(3)

11123	4123	9123
6123	8123	10123
7123	12123	5123

(4)

$(n+7)\,10^M + a_{n-1}$	$n10^M + a_{n-1}$	$(n+5)10^M + a_{n-1}$
$(n+2)10^M + a_{n-1}$	$(n+4)10^M + a_{n-1}$	$(n+6)10^M + a_{n-1}$
$(n+3)10^M + a_{n-1}$	$(n+8)10^M + a_{n-1}$	$(n+1)10^M + a_{n-1}$

(5)

Fig. 5. Constructing Magic squares 3x3 from k-truncated numbers of S_5-series.

6. Magic squares 3×3 in size from k-truncated numbers of S_6-series. Numbers of S_6-series {see (18)} resemble both numbers of the first subsequence of S_2-series and numbers of S_3-series {see points 2 and 3}.The example of the Magic square 3×3 from numbers of S_6-series is presented in Fig. 6. This square is constructed by means of methods described in points 1 – 3. Thus, in spite of the mentioned difference between numbers of S_6-series and S_2-, S_3-series for solving problems on constructing Magic square 3×3 from numbers of S_6-series the methods, discussed above, can be applied.

2527293132302826	1113151718161412	2123252728262422
1517192122201816	1921232526242220	2325272930282624
1719212324222018	2729313334323028	1315171920181614

Fig. 6. Constructing Magic squares 3×3 from *k*-truncated numbers of S_6-series.

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