ADVANCE OF SMARANDACHE APPROACH TO SOLVING SYSTEMS OF DIOPHANTINE EQUATIONS

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By developing F. Smarandache (algebraic) approach to solving systems of Diophantine equations we elaborate a set of new computative algorithms and analytical formulae, which may be used for finding numerical solutions of some combinatorial and number-theoretic problems.

Key words: systems of Diophantine equations, algebraic approach, combinatorics, number theory, Magic and Latin squares.

1 Introduction

Let it be required to solve some system of Diophantine equations. In this case [1] algebraic methods can be applied for

a) constructing the total algebraic solution of the system;

b) finding the transformations translating an algebraic solution of the system from one form into another one;

c) elucidating the general legitimacies existing between the elements of the algebraic solution;

d) replacing the total algebraic solution containing L arbitrary selected parameters by a set of algebraic solutions containing less than L parameters.

This paper is devoted to further advance of algebraic approach to solving systems of Diophantine equations. In particular, in this investigation we

1) describe the simple way of obtaining a total solution of systems of Diophantine linear equations in the integer numbers, and show (see Sect. 2) that this way may be considered as some modification of F. Smarandache algorithm 3 from his work [2];

2) demonstrate the effectiveness of the algebraic approach to the elaboration of computative algorithms and analytical formulae, which may be used respectively for obtaining the required numerical solutions of the discussed systems and for counting of the total quantity of solutions from a given class of numbers (Sect. 3);

3) derive analytical formulae available for constructing classical Magic squares of both odd and even orders (Sect. 4).

2 The way of obtaining the total solution of systems of Diophantine linear equations in the integer numbers

Let it be required to solve some system of linear Diophantine equations in the integer numbers. It seems to be evident, that there is no complication in solving this problem at present. For instance, one may find in the work[2] as many as five different algorithms to obtain a total solution of this problem, which correctness are proved by mathematical methods and illustrated by concrete examples. In particular, to illustrate the correctness of an algorithm 3, the system from three following equations

$$\begin{cases} 3x_1 + 4x_2 \quad + \quad 22x_4 - 8x_5 = 25 \\ 6x_1 \quad + \quad 46x_4 - 12x_5 = 2 \\ \quad 4x_2 + 3x_3 - x_4 + 9x_5 = 26 \end{cases} \tag{1}$$

are solved in the work[2]. The final solution, obtained for (1) by the mentioned algorithm, has form

$$x_1 = -40k_1 - 92k_2 + 27; \quad x_2 = 3k_1 + 3k_2 + 4; \quad x_3 = -11k_1 + 8; \tag{2}$$
$$x_4 = 6k_1 + 12k_2 - 4; \quad x_5 = 3k_1 - 2,$$

where k_1 and k_2 are any integer numbers.

Let us clear up a question whether (2) is the total solution of system (1) in the integer numbers. To make it we will solve (2) by the algebraic methods with testing their correctness on every step of our computations.

1. As well-known[1-3], the total algebraic solution of the system (1) may be found by standard algebraic methods (for instance, by Gauss method). In our case it has the form

$$\begin{cases} x_1 = -(23x_4 - 6x_5 - 1)/3 \\ x_2 = (x_4 + 2x_5 + 24)/4 \\ x_3 = (-11x_5 + 2)/3 \end{cases} \tag{3}$$

that coincides with the solution found on the first step of the algorithm 3 of the work[2].

2. As well-known from the theory of comparison[3,4], the total solution in the integer numbers for the last equation of the system (3) has the form $x_3 = \{-11(3m_1 + m_0) + 2\}/3$, where m_1 is any integer number; the value of m_0 is equal 0, ± 1 or ± 2 and is chosen from the condition that the number $(-11m_0 + 2)/3$ must be integer. Thus, we find on second step of our computations that $x_5 = 3m_1 + 1$ and $x_3 = -11m_1 - 3$.

We note that the solution $x_5 = 3k_1 - 2$ of (2) may be obtained from our solution by change of the variable m_1 to $k_1 - 1$. Thus, both values of x_5 are identical solutions.

3. Let us get to solving the second equation of the system (3) in the integer numbers. Replacing the value of x_5 by $3m_1 + 1$ in this equation we obtain that $x_2 = 6 + (6m_1 + x_4 + 2)/4$. Hence it appears (see point 2) that $x_4 = (-2 + 4l_1)m_1 - 2 + 4l_2$, where l_1 and l_2 are any integer numbers.

4. Replacing the value of x_4 by $(-2 + 4l_1)m_1 - 2 + 4l_2$ in the first equation of the system (3) we obtain that $x_1 = 2x_5 - \{23(-2 + 4l_1)m_1 + 92l_2 - 47\}/3$. Hence it appears (see points 2 and 3) that $l_1 = 3m_2 + 2$ and $l_2 = 3m_3 + 1$ and, consequently, the total solution of the system (1) in the integer numbers has the form

$$x_1 = -4m_1(23m_2 + 10) - 92m_3 - 13; \quad x_2 = 3m_1(m_2 + 1) + 3m_3 + 7; \tag{4}$$
$$x_3 = -11m_1 - 3; \quad x_4 = 6m_1(2m_2 + 1) + 2(6m_3 + 1); \quad x_5 = 3m_1 + 1,$$

where m_1, m_2 and m_3 are any integer numbers.

Comparing (4) with (2) we find that

a) the solution (4) contains greater by one parameter than solution (2);

b) if $m_2 = 0$, $m_1 = k_1 - 1$ and $m_3 = k_2$ in the solution (4) then the solution (4) coincides with the solution (2).

Thus, the solution (4) contains all numerical solutions of the system (1), which may be obtained from (2), but a part of numerical solutions, which may be obtained by (4), can not obtain from (2) or, in other words, (2) is not the total solution of the system (1) in the integer numbers.

We add that, in general, a partial loss of numerical solutions of systems of linear Diophantine equations may have more serious consequences than in the discussed case. For instance, as it has been proved in the work[5] by using the algebraic approach to solving systems of Diophantine equations,

if Magic squares of 4th order contain in its cells 8 even and 8 odd numbers then they can not have structure patterns another than 12 ones, adduced in works[1, 3, 5–7] *for Magic squares, contained integer numbers from 1 to 16.*

In reality, this statement is incorrect because yet several new structure patterns may exist for Magic squares from 8 even and 8 odd numbers[1, 3].

3 Analysing a system from 8 linear Diophantine equations

To demonstrate the effectiveness of the algebraic approach to solving some combinatorial and number-theoretic problems, presented in the form of systems of Diophantine equations, in this section we will analyse the following system from 8 linear Diophantine equations

$$\begin{array}{lll} 1.\ a_1 + a_2 + a_3 = S, & 4.\ a_1 + a_4 + a_7 = S, & 7.\ a_1 + a_5 + a_9 = S, \\ 2.\ a_4 + a_5 + a_6 = S, & 5.\ a_2 + a_5 + a_8 = S, & 8.\ a_3 + a_5 + a_7 = S. \\ 3.\ a_7 + a_8 + a_9 = S, & 6.\ a_3 + a_6 + a_9 = S, & \end{array} \tag{5}$$

We note if symbols $a_1, a_2, \ldots, a_9$ are arranged as in the table 1, shown in figure, and their values are replaced by ones, which are taken from some total algebraic solution of the system (5), then table 1 will be transformed into the total algebraic formula of Magic squares of 3rd order. In other words, the discussed problem on solving the system (5) connects direct with the well-know ancient mathematical problem on constructing numerical examples of Magic squares of 3rd order.

3.1 Requirements to a set of numbers, which is the solution of the system (5)

Proposition 1. *A set of nine numbers is a solution of the system (5) only in the case if one succeeds to represent these nine numbers in the form of such three arithmetic* *progressions from 3 numbers whose differences are identical and the first terms of all three progressions are also forming an arithmetic progression.*

Proof. Using standard algebraic methods (for instance, Gauss method) we find that the total algebraic solution of the system (5) has the form

$$a_1 = 2a_5 - a_9;\quad a_2 = 2a_9 + a_6 - 2a_5;\quad a_3 = 3a_5 - a_6 - a_9;\quad a_4 = 2a_5 - a_6;\quad a_7 = a_9 + a_6 - a_5;\quad a_8 = 4a_5 - 2a_9 - a_6, \tag{6}$$

where values of parameters a_5, a_6 and a_9 are chosen arbitrarily. Arranging solutions (4) in order, shown in the table 2 (see figure), we obtain the table 3. It is noteworthy that arithmetic progressions with the difference $2a_5 - a_6 - a_9$ place in the rows of the table 3, whereas ones, having the difference $a_5 - a_9$, place in its columns. If one introduces three new parameters a, b and c by the equalities $a_5 = a + b + c$, $a_6 = a + 2c$ and $a_9 = a + b$ into the table 3, then this table will acquire more elegant form, which it has in table 4, and so the fact of existing of the arithmetic progressions in it will receive more visual impression. Thus, the proof of Proposition 1 follows directly from the construction of tables 3 and/or 4 and it is appeared as a result of using the algebraic methods, mentioned in the points (a) and (c) of Sect. 1.

a_1	a_2	a_3
a_4	a_5	a_6
a_7	a_8	a_9

(1)

$2a_9 + a_6 - 2a_5$	a_9	$2a_5 - a_6$
$a_9 + a_6 - a_5$	a_5	$3a_5 - a_6 - a_9$
a_6	$2a_5 - a_9$	$4a_5 - 2a_9 - a_6$

(3)

a_2	a_9	a_4
a_7	a_5	a_3
a_6	a_1	a_8

(2)

a	$a + b$	$a + 2b$
$a + c$	$a + b + c$	$a + 2b + c$
$a + 2c$	$a + b + 2c$	$a + 2b + 2c$

(4)

Figure. Elucidating the general legitimacies existing between the elements of the solution (6).

3.2 Elaboration of a universal algorithm for finding all numerical solutions of the system (5) from a given class of numbers

Let it be required to find all numerical solutions of the system (5), which belong to the given class of numbers and has $a_5 = f$. For elaboration of a universal algorithm, solving this problem, we first write out all possible decompositions of the number $2f$ in the two summands of the following form

$$2f = x_1(j) + x_2(j), \tag{7}$$

where j is the number of a decomposition; $x_1(j)$ and $x_2(j)$ are two such numbers that $x_1(j) < x_2(j)$ and both ones belong to the given class of numbers. In a complete set of various decompositions of the kind, we fix only one, having, for instance, the number k. Determine for it the number $d(k)$

$$d(k) = 2f - x_1(j). \tag{8}$$

Proposition 2. *The desirable numerical solution of the system (5), which contains $a_5 = f$ and numbers $x_1(k)$ and $x_2(k)$ of (7), can be found only in the case, if one succeeds to find, among the remaining numbers of the form $x_1(k)$, an arithmetic progression from three numbers with the difference $d(k)$.*

Proof. The truth of Proposition 2 follows from the construction of the tables 3 and/or 4, shown in figure.

It is evident also, to obtain a complete set of solutions of the system (5) from the given class of numbers, one should repeat the foregoing actions for all the differences $d(k)$.

3.3 Deriving an analytical formula for counting the quantity of various solutions of the system (5) from natural numbers

Proposition 3. *If $A(m)$ is the total number of various solutions of the system (5) from natural numbers and $a_5 = m$ then its value may be computed by the formula*

$$A(m) = 9[m/6]^2 + \{3\,(m \bmod 6) - 8\}[m/6] + 2 - 2\,[\{(m \bmod 6) + 5\}/6] + [(m \bmod 6)/5]. \tag{9}$$

Proof. We first write out all possible decompositions of the number $2m$ in two distinct terms.

$$\begin{array}{lll}
2m = 1 + (2m-1) & (j = 1, & d(1) = m-1), \\
2m = 2 + (2m-2) & (j = 2, & d(2) = m-2), \\
\dots\dots\dots & \dots\dots\dots & \\
2m = m-2 + (m+2) & (j = m-2, & d(m-2) = 2), \\
2m = m-1 + (m+1) & (j = m-1, & d(m-1) = 1).
\end{array} \tag{10}$$

The problem on counting total number of various solutions of the system (5) with $a_5 = m$ is now reduced, in accordance with the universal algorithm of Sect. 3.2, to counting a total number of various arithmetic progressions consisting of three numbers, which may be composed from the numbers $1, 2, \dots, m-2, m-1$ and such that the differences in these progressions are respectively equal to $d(m-1), d(m-2), \dots, d(1)$.

To simplify this new problem we shall deduce a recurrence relation which will link the total numbers of various solutions having $a_5 = m$ and $a_5 = m - 1$. For this aim we decompose all the solutions with $a_5 = m$ in two groups. The solutions, having number 1, will be attributed to the first group. A total number of such solutions will be denoted by $A_1(m)$. All the remaining solutions we shall attribute to the second group. We decrease now each number by 1 in all solutions of the second group. After this operation, a lot of the second group solutions will represent by themselves a complete set of various solutions from natural numbers with $a_5 = m - 1$. Thus, the following relation

$$A(m) = A_1(m) + A(m-1) \tag{11}$$

is valid or, in other words, if we know a value of $A(m-1)$ then for finding the value $A(m)$ it will be sufficient to count the number of the solutions containing $a_5 = m$ and the number 1. This new combinatorial problem can be reformulated as the following one

to find a total number of various arithmetic progressions from three numbers which can be composed from the sequence of numbers 1, 2, ..., $m-2$, $m-1$ and such that the first number of these progressions is number 1 and the differences of the progressions are respectively equal to $d(m-1)$, $d(m-2)$, ..., $d(1)$.

It seems to be evident, that a total number of the desired progressions coincides with the maximal difference value of the progression D_{max} for which one can still find an arithmetic progression of the required form from the set of numbers 1, 2, ..., $m-2$, $m-1$. The value of D_{max} can be found from the correlation $1 + 2D_{max} = m-1$, whence $D_{max} = [(m-2)/2]$, where square brackets denote the integer part. But in reality this value of D_{max} is not always coinciding with the value of $A_1(m)$: to eliminate this non-coincidence we must decrease the total number of arithmetic progressions by one if numbers $1 + d(k)$ or $1 + 2d(k)$ coincide with the number $x_1(k)$ of (7).

Let us determine at which values of $d(k)$ this coincidence occurs:

$$1 + d(k_1) = k_1 = m - d(k_1); \quad 1 + 2d(k_2) = k_2 = m - d(k_2), \tag{12}$$

whence $d(k_1) = (m-1)/2$, and $d(k_2) = (m-1)/3$. If $d(k_1) = (m-1)/2$ the number $1 + 2d(k_1) > m-1$. Consequently, this case is never fulfilled. The coincidence occurs in the second case if $m-1$ is multiple of 3.

If we decompose all m-numbers in six classes so that the numbers of the form $6k$ will be attributed to the first class and those of the form $6k+1$ — to the second one and so on, where $k = 1, 2, \ldots$, then for all six classes of the m-numbers one can write out in the explicit form the values of D_{max} and $A_1(m)$:

$$\begin{array}{lll}
m = 6k, & D_{max} = 3k-1, & A_1(6k) = 3k-1; \\
m = 6k+1, & D_{max} = 3k-1, & A_1(6k+1) = 3k-2; \\
m = 6k+2, & D_{max} = 3k, & A_1(6k+2) = 3k; \\
m = 6k+3, & D_{max} = 3k, & A_1(6k+3) = 3k; \\
m = 6k+4, & D_{max} = 3k+1, & A_1(6k+4) = 3k; \\
m = 6k+5, & D_{max} = 3k+1, & A_1(6k+5) = 3k+1.
\end{array} \tag{13}$$

Further we shall need the value of the difference $\Delta A(k, i)$ of the following form

$$\Delta A(k, i) = A(6(k+1)+i) - A(6k+i), \tag{14}$$

where $i = 0, 1, \ldots, 5$. Using (14), (13) and (11) we may find an explicit expressions for $\Delta A(k, i)$. Let, for instance, $i = 0$. Then

$$\begin{aligned}
\Delta A(k, 0) = A(6(k+1)) - A(6k) &= A_1(6(k+1)) + A_1(6k+5) + \\
&+ A_1(6k+4) + A_1(6k+3)) + A_1(6k+2) + A_1(6k+1) = \\
&= \{3(k+1)-1\} + (3k+1) + 3k + 3k + 3k + (3k-2) = 18k+1.
\end{aligned} \tag{15}$$

Remaining values of $\Delta A(k, i)$ for $i = 0, 1, \ldots, 5$ can be found analogously:

$$\Delta A(k, i) = 18k + 1 + 3i. \tag{16}$$

It is evident, since the $\Delta A(k, i)$ is a linear function from k, values $A(6k + i)$ may be obtained from the second degree polynomial

$$A(6k + i) = b_2(i)\, k^2 + b_1(i)\, k + b_0(i), \tag{17}$$

where $b_2(i) = 9$, $b_1(i) = 3i - 8$, $b_0(i) = 2 - 2[(i + 5)/6] + [i/5]$; square brackets mean the integer part. Taking into account that $i = (m \bmod 6)$, $k = [m/6]$ and $6k + i = m$ we may obtain (9) from (17).

It should be noted that using regression analysis methods one may appreciably simplify [8, 9] the expression (9):

$$A(m) = g\{(3m^2 - 16m + 18.5)/12\}, \tag{18}$$

where the notation $g\{a\}$ means the nearest integer to a.

4 Algebraic approach to deriving analytical formulae available for constructing classical Magic squares of the n-th order

We remind that in the general case [1, 3] *Magic* square*s* represent by themselves numerical or analytical square tables, whose elements satisfy a set of definite basic and additional relations. The basic relations therewith assign some constant property for the elements located in the rows, columns and two main diagonals of a square table, and additional relations, assign additional characteristics for some other sets of its elements. In particular, when the constant property is a significance of sum of various elements in rows, columns or main diagonals of the square, then this square is an *Additive* one. If an Additive square is composed of successive natural numbers from 1 to n^2, then it is a *Classical* one.

It is evident [3, 10, 11] that, from the point of view of mathematics, the analytical solution of the problem on constructing Classical squares of the n-th order consists of determining a form of f and g functions, which permit to compute the position for any natural number N from 1 to n^2 in cells of these squares: $x = f(N, n)$ and $y = g(N, n)$.

In this section we

1) adduce two types of analytical functions, by which one may construct Classical squares of odd orders;

2) reveal a connection between these analytical functions and Latin squares;

3) give an algebraic generalisation of the notion "Latin square";

4) derive analytical formulae available for constructing Classical squares of both odd and even orders.

4.1 Classical approach to deriving analytical formulae available for constructing Magic squares of odd order from natural numbers

For any linear algorithmic methods of constructing Classical squares of odd orders, the functions f and g have the following forms [3, 10, 11]:

$$f(N,\ n) \equiv a_1(N-1) + b_1[\,(N-1)/n\,] + c_1 \pmod n, \qquad (19)$$
$$g(N,\ n) \equiv a_2(N-1) + b_2[\,(N-1)/n\,] + c_2 \pmod n,$$

where square brackets mean the integer part; a sign "$\equiv$" is the modulo n equality; N is any natural number from 1 to n^2; a_1, b_1, c_1 and a_2, b_2, c_2 are such integral coefficients, that the numbers a_1, a_2, b_1, b_2; $a_1b_2 - a_2b_1$; $a_2 - a_1$, $b_2 - b_1$, $a_2 + a_1$ and $b_2 + b_1$ are mutually disjoint with n.

There is no difficulty in counting that in formula (19) the coefficients $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ are equal[3, 10, 11]

— $\{$ 1, 1, $-[n/2]$; 1, -1, $[n/2]$ $\}$ for Terrace algorithmic method of constructing Classical squares;

— $\{1, -1, n/2]; 1, -2, n-1\}$ for Siamese method;

— $\{1, -1, [n/2];\ 2, 2, 0\}$ for Knight method;

— $\{\ (3-a)q + (a+1)/2,\ (3-a)q + (a-1)/2,\ 0;\ \ (3-a)q + (a-1)/2,\ (3-a)q + (a+1)/2,\ 0\ \}$ (where $q = [(n+1)/6]$, $a = n - 6q$) for the classical square of the n-th order, which, if n is an odd number, non-divisible by three, can be formed also from a pair of orthogonal Latin squares, constructed by the pair of comparisons $x + 2y \pmod n$ and $2x + y \pmod n$; and so on.

It should be noted, that the above conditions for coefficients of functions f and g become contradictory for even n. For example, by the conditions, the coefficients a_1, a_2, b_1 and b_2 of the functions f and g of (19) should be mutually disjoint with n, and consequently, if n is even, they must be odd. The same requirement must be the true for the number $d = a_1b_2 - a_2b_1$. But if a_1, a_2, b_1 and b_2 are odd, the number d, which is the difference of the two odd numbers, will be an even number.

Thus, an essential fault of linear formulae of (19) is the impossibility of using them for constructing Classical squares of even orders.

4.2 Revealing a connection between Latin squares and analytical formulae of (19)

Proposition 4. *If a Classical square of the n-th order is constructed by formulae (19), then it may be constructed also by the formula*

$$N(x, y) = n\,p(x, y) + r(x, y) + 1, \qquad (20)$$

where $p(x, y) \equiv \alpha_1 x + \beta_1 y + \sigma_1$ *and* $r(x, y) \equiv \alpha_2 x + \beta_2 y + \sigma_2$.

Proof. The equivalence of formulae (19) and (20) appears from their linearity and the fact, that (20) are inverse formulae to (19). In particular, if values of coefficients $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ of formulae (19) are known then values of $\{\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2\}$ of formulae (2) may be computed from following linear equations[11]

$$m\alpha_1 \equiv -a_2;\ m\beta_1 \equiv a_1;\ m\sigma_1 \equiv a_2c_1 - a_1c_2; \qquad (21)$$
$$m\alpha_2 \equiv b_2;\ m\beta_2 \equiv -b_1;\ m\sigma_2 \equiv b_1c_2 - b_2c_1;$$
$$m = a_1b_2 - a_2b_1$$

and, reciprocally, at the reverse task, the values of $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ may be computed from equations:

$$\mu a_1 \equiv -\beta_1;\ \ \mu b_1 \equiv \beta_2;\ \ \mu c_1 \equiv \beta_1\sigma_2 - \beta_2\sigma_1; \tag{22}$$
$$\mu a_2 \equiv \alpha_1;\ \ \mu b_2 \equiv -\alpha_2;\ \ \mu c_2 \equiv \alpha_2\sigma_1 - \alpha_1\sigma_2;$$
$$\mu = \alpha_1\beta_2 - \alpha_2\beta_1.$$

For instance, values of $\{\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2\}$ of formulae (20) are equal

— $\{(n+1)/2, (n-1)/2, (n-1)/2; (n+1)/2, (n+1)/2, 0\}$ for Terrace method;

— $\{1, n-1, (n-1)/2;\ 2, n-1, (n-1)/2$ \} for Siamese method;

— $\{[n/2], a, n-a; (n+1)/2, a, a\}$ for Knight method, where $a = dc + (n-d)(1-c)$, $d = [(n+1)/4]$, $c = [(n \bmod 4)/2]$.

We remind[3, 12] that, a quadratic table $n{\times}n$ in size is *Latin* square of n-th order if only n elements of this table are different and each of these n elements occurs only one time in each row and column of the table. The two Latin squares P and R of the same order n are called *orthogonal* if all the pairs, formed by their elements p_{ij} and r_{ij} (i is the number of a row; j is the number of a column) are different.

Proposition 5. *If elements of a Latin square of the n-th order are numbers 0, 1, ..., n − 1, then, for constructing such Latin square, one may use a linear comparison*

$$L(x, y) \equiv \alpha x + \beta y + \sigma, \tag{23}$$

where α and β are integer numbers, which are to be mutually disjoint with n; σ is any integer number.

Proof. Let the numbers $L(x, y)$ of (23) are located in each cells of a quadratic table $n{\times}n$ in size. We consider n numbers, which are located in row y_0 of this table. Since the discussed numbers are obtained from the linear comparison (23) at $x = 0, 1, \ldots, n-1$, to show that all they are different, we should demonstrate that they belong to different modulo n classes. Let $x_1 > x_2$ and $\alpha x_1 + \beta y_0 + \sigma \equiv \alpha x_2 + \beta y_0 + \sigma$. Since $\beta y_0 + \sigma$ is a constant, in accordance with the properties of comparisons[3, 4], we obtain the new equality $\alpha x_1 \equiv \alpha x_2$. Hence, since α is mutually disjoint with n, $x_1 \equiv x_2$. But this equality contradicts our assumption. Thus, each of numbers $0, 1, \ldots, n-1$ occurs only one time in each row and column of the discussed table and so this table is Latin square of n-th order.

Proposition 6. *Every Classical square of the odd order, decomposed on two orthogonal Latin squares, may be constructed by the formulae (19) and otherwise.*

Proof. The truth of Proposition 6 follows directly from Propositions 4 and 5 and conditions for coefficients of functions f and g of (19).

4.3 Deriving analytical formulae available for constructing Classical squares of both odd and even orders

The way 1. Let us give an algebraic generalisation of the notion "Latin square":

a quadratic table n×n in size is the generalised Latin square of n-th order if only n elements of this table are different and each of these n elements occurs only n times in this table.

Proposition 7. *Every Classical square of a order n may be decomposed on two orthogonal generalised Latin squares P and R of the order n.*

Proof. To prove Proposition 7, it is sufficient to note that

a) any integer number N from 1 to n^2 may be presented in the form

$$N = np + r + 1, \tag{24}$$

where p and r can take values only $0, 1, \ldots, n-1$;

b) each of values $0, 1, \ldots, n-1$ of parameters p and r occurs n times precisely in the decomposition (24) of numbers N.

Thus, to construct two orthogonal generalised Latin squares P and R from a Classical square of a order n, one should replace in the Classical square all numbers N by respectively $(N-1) \bmod n$ and $[(N-1)/n]$.

Proposition 8. *Every Classical square of order n may be constructed by the formula (20), in which functions p(x, y) and r(x, y) may belong, in general case, to both linear and non-linear classes of ones.*

Proof. The truth of Proposition 8 follows directly from Propositions 7 and materials of Sect. 4.1.

We note, in particular, one may construct Classical squares of even-even orders n ($n = 4k$; $k = 1, 2, \ldots$) by the analytical formula (20), in which functions $p(x, y)$ and $r(x, y)$ have the following forms [3, 11, 13]

$$p(x, y) = cx + (1-c)(n-x-1) \text{ and } r(x, y) = (1-c)\,y + c\,(n-y-1), \tag{25}$$

where $c = \{[(x+1)/2] + [(y+1)/2]\} \bmod 2$; or

$$p(x, y) = cd - x - 1 + (1-c)(n-d) \text{ and } r(x, y) = by + (1-b)(n-y-1), \tag{26}$$

where $c = (x + y + a) \bmod 2$; $d = (1-a)\,y + a\,(n-y-1)$; $b = \{[(x+3)/2] + [y/2] + a\} \bmod 2$; $a = [2y/n]$; and so on.

The way 2. It is evident, we may consider Classical squares not only as the sum of two orthogonal generalised Latin squares (see the way 1) but, for instance, as quadratic tables whose rows contain certain numerical sequences. Let us look into the problem on finding universal analytical formulae for constructing Classical squares from this new point of view.

Proposition 9. *If a Classical square of the n-th order is constructed by formulae (19), then it may be constructed also by the formula*

$$N(x, y) \equiv a + b - \lambda c, \tag{27}$$

where a, b and c are any integer numbers; λ is 0 or 1, the sign "≡" is the modulo n^2 equality.

Proof. Let a Classical square of an odd order is constructed by formulae (19). It follows from Proposition 4 that this square may be constructed also by formulae (20). We deduct x-th element of first row from every x-th element of all y-th rows of the Classical square. It is evident that the number

$$\langle\ n\{(\alpha_1 x + \beta_1 y + \sigma_1) \bmod n\} \ - \ n\{(\alpha_1 x \ + \sigma_1) \bmod n\} + \\ (\alpha_2 x \ + \beta_2 y + \sigma_2) \bmod n \ - \ (\alpha_2 x \ + \sigma_2) \bmod n \ \rangle \bmod n^2. \tag{28}$$

will be located in the cell (x, y) of the reformed Classical square {see (20)}. Using the equality $(dn) \bmod n^2 = n\,(d \bmod n)$ we present (28) as the sum of two summands

$$n\{(\beta_1 y)\} \bmod n + \{(\alpha_2 x + \beta_2 y + \sigma_2) \bmod n - (\alpha_2 x + \sigma_2) \bmod n\}. \tag{29}$$

The second summand of (29) may have only two values: $(\beta_2 y) \bmod n$ or $n - (\beta_2 y) \bmod n$. Thus, we obtain that numbers of any y-th row of the reformed Classical square may have only two values. By using the mentioned method of constructing formula (27), we find that parameters of this formula a, b, c and λ are connected with parameters of the formula (20) by correlations

$$a = n\{(\alpha_1 x + \sigma_1) \bmod n\} \ + \ (\alpha_2 x + \sigma_2) \bmod n, \tag{30}$$

$$b = n\{(\beta_1 y) \bmod n\} + \ (\beta_2 y) \bmod n, \quad c = n,$$

$$\lambda = [\ 1 - \ \mathrm{sign}\{(\alpha_2 x \ + \beta_2 y + (64) \bmod n - \ (\alpha_2 x + (64) \bmod n\ \}\ ]/2,$$

where $\mathrm{sign}(x) = |x|/x$ if $x \neq 0$ and $\mathrm{sign}(0) = 0$.

It should be noted, if we get off the sign "$\equiv$" in the formula (27) and translate correlations (30) into language of numerical sequences {see the point (c) of Sect. 1}, we obtain that, for algorithmic methods which mentioned in Sect. 4.1 and 4.2, the parameters of formula (23) are determined by correlations

$$a = -(-1)^k n(n-1)/4 + k(n+1)/2 + \{n(n-3) + 2\}/4, \tag{31}$$

$$b = n - 1 - y, \quad c = n, \quad \lambda = [\ \mathrm{sign}\{(h - y) + 2\}/2\ ],$$

$$h = [\ z/2\ ] - 1 + (n+1)\,(\,z/2 - [\ z/2\ ]\,),$$

$$\sigma_y(z) = y + z + 2 - \ n\,[\ \mathrm{sign}\{\ (y + z - n + 1) + 2\ \}/2\ ], \quad k_1 = \sigma_y(x)$$

where the numerical sequence $\{a_k\}$, if its values are computed at $k = 0, 1, \ldots, n-1$, coincides with the numerical sequence, located in the first row of the Classical square; $\sigma_y(t)$ is a permutation operator of numbers $0, 1, \ldots, n-1$ and for

— Terrace method $k = k_1$, $z = k_1 - 1$;

— Siamese method $z \equiv \{n - 2(k_1 - 1)\} \bmod n$, $k = z + 1$;

— Knight method $k_2 = \sigma_y(k_1 - 1)$, $z = -(-1)^{k_2} n/4 + k_2/2 + (n+4)/4$, $k = z + 1$.

It is evident, using the formula (27) with parameters (31), one can have no difficulty in discovering "genetic connections" between different Classical squares and constructing methods and in generating a set of new methods. For instance, if n is an odd number, non-divisible by three, the new algorithmic methods for constructing Classical squares of odd orders appear when $k_1 = \sigma_y^{\,4}(x)$ or $k_1 = \sigma_y^{\,7}(x)$ in (31), or the form of σ_y and/or the numerical sequence $\{a_k\}$ is changed.

It remains for us to add that parameters of the formula (27) are determined by correlations

$$a = nk, \quad b = w, \quad c = n - 2w - 1, \quad \lambda = [((k+2) \bmod 4)/2], \tag{32}$$

$$\sigma_w(z) = 1 + \{\ z \ - \ h\,(\,2\,(\,(z + h) \bmod 2\,) - 1\,)\ \} \ \bmod n,$$

$h = y + c\,[\,y/[\,y/2\,]\,]$, $\quad k_1 = \sigma_w(z)$

for formulae (25) and (26), where $\sigma_w(z) = 1 + \{z - h\,(2((z + h) \bmod 2) - 1)\} \bmod n$; $h = y + c\,[\,y/[\,y/2]\,]$; $\quad k_1 = \sigma_w(z)$ and for

— the formula (26) $z = x$; $\quad w = y$; $\quad k = k_1$;

— the formula (25) $\lambda_y = [((\,y + 1) \bmod 4\,)/2]$; $\quad k = k_1$; $\quad \lambda_x = [((x+1+2\lambda_y) \bmod 4)/2]$; $\quad z \equiv (x_s + n - h\,(1 - 2(x_s \bmod 2))) \bmod n$; $\quad x_s = \lambda_x x + (1 - \lambda_x)\,(\,n - x - 1)$; $w = \lambda_y\,y + (1 - \lambda_x)\,(\,n - y - 1)$.

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