ON THE CUBIC RESIDUES NUMBERS AND k-POWER COMPLEMENT NUMBERS

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ABSTRACT. The main purpose of this paper is to study the asymptotic property of the the cubic residues and k-power complement numbers (where $k \geq 2$ is a fixed integer), and obtain some interesting asymptotic formulas.

1. INTRODUCTION AND RESULTS

Let a natural number $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then $a_3(n) = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_r^{\beta_r}$ is called a cubic-power residues number, where $\beta_i = \min(2, \alpha_i), 1 \leq i \leq r$; Also let $k \geq 2$ is a fixed integer, if $b_k(n)$ is the smallest integer that makes $nb_k(n)$ a perfect k-power, we call $b_k(n)$ as a k-power complement number. In problem 64 and 29 of reference [1], Professor F. Smarandache asked us to study the properties of the cubic residues numbers and k-power complement numbers sequences. By them we can define a new number sequences $a_3(n)b_k(n)$. In this paper, we use the analytic method to study the asymptotic properties of this new sequences, and obtain some interesting asymptotic formulas. That is, we shall prove the following four Theorems.

Theorem 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} a_3(n)b_k(n) = \frac{6x^{k+1}}{(k+1)\pi^2} R(k+1) + O\left(x^{k+\frac{1}{2}+\varepsilon}\right),$$

where ε denotes any fixed positive number, and

$$R(k+1) = \prod_p \left(1 + \frac{p^3+p}{p^7+p^6-p-1}\right)$$

if $k = 2$ and

$$R(k+1) = \prod_p \left(1 + \sum_{j=2}^{k} \frac{p^{k-j+3}}{(p+1)p^{(k+1)j}} + \sum_{j=1}^{k} \frac{p^{k-j+3}}{(p+1)(p^{(k+1)(k+j)} - p^{(k+1)j})}\right)$$

if $k \geq 3$.

Key words and phrases. cubic residues numbers; k-power complement numbers; Asymptotic formula; Arithmetic function .

Theorem 2. *Let $\varphi(n)$ is the Euler function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x}\varphi(a_3(n)b_k(n)) = \frac{6x^{k+1}}{(k+1)\pi^2}R^*(k+1) + O\left(x^{k+\frac{1}{2}+\varepsilon}\right)$$

where

$$R^*(k+1) = \prod_p\left(1+\frac{p^2+1}{p^6+2p^5+2p^4+2p^3+2p^2+2p+1}-\frac{1}{p^2+p}\right)$$

if $k=2$, and

$$R^*(k+1) = \prod_p\left(1-\frac{1}{p^2+p}+\sum_{j=2}^{k}\frac{p^{k-j+3}-p^{k-j+2}}{(p+1)p^{(k+1)j}}+\sum_{j=1}^{k}\frac{p^{k-j+3}-p^{k-j+2}}{(p+1)(p^{(k+1)(k+j)}-p^{(k+1)j})}\right)$$

if $k \geq 3$.

Theorem 3. *Let $\alpha > 0$, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x}\sigma_\alpha(a_3(n)b_k(n)) = \frac{6x^{k\alpha+1}}{(k\alpha+1)\pi^2}R(k\alpha+1) + O\left(x^{k\alpha+\frac{1}{2}+\varepsilon}\right),$$

where

$$R(k\alpha+1) = \prod_p\left(1+\frac{p}{p+1}\left(\frac{p^\alpha+1}{p^{2\alpha+1}}+\frac{(p^{3\alpha}-1)p^{2\alpha+1}+p^{4\alpha}-1}{(p^{3(2\alpha+1)}-p^{2\alpha+1})(p^\alpha-1)}\right)\right)$$

if $k=2$, and

$$R(k\alpha+1) = \prod_p\left(1+\frac{p^{k\alpha+1}-p}{(p+1)(p^\alpha-1)p^{k\alpha+1}}+\sum_{j=2}^{k}\frac{p^{(k-j+3)\alpha+1}-p}{(p+1)(p^\alpha-1)p^{(k\alpha+1)j}}\right.$$
$$\left.+\sum_{j=1}^{k}\frac{p^{(k-j+3)\alpha+1}-p}{(p+1)(p^\alpha-1)(p^{(k+j)(k\alpha+1)}-p^{(k\alpha+1)j})}\right)$$

if $k \geq 3$.

Theorem 4. *Let $d(n)$ denotes Dirichlet divisor function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x}d(a_3(n)b_k(n)) = \frac{6x}{\pi^2}R(1)\cdot f(\log x) + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $f(y)$ is a polynomial of y with degree k. and

$$R(1) = \prod_p\left(1+\frac{p^3}{(p+1)^3}\left(\frac{3p+4}{p^3+p}-\frac{3}{p^2}-\frac{1}{p^3}\right)\right)$$

if $k=2$, and

$$R(1) = \prod_p\left(1+\sum_{j=2}^{k}\frac{\left(k-j+3-\binom{k+1}{j}\right)p^{k-j+1}}{(p+1)^{k+1}}+\sum_{j=1}^{k}\frac{k-j+3}{(p+1)^{k+1}(p^{j-1}-p^{j-k-1})}-\frac{1}{(p+1)^{k+1}}\right)$$

if $k \geq 3$.

2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. Let

$$f(s)=\sum_{n=1}^{\infty}\frac{a_3(n)b_k(n)}{n^s}.$$

From the Euler product formula [2] and the definition of $a_3(n)$ and $b_k(n)$ we have

$$\begin{aligned} f(s)&=\prod_p\left(1+\frac{a_3(p)b_k(p)}{p^s}+\frac{a_3(p^2)b_k(p^2))}{p^{2s}}+\cdots\right)\\ &=\prod_p\left(1+\frac{1}{p^{s-2}}+p^2(\frac{1}{p^{2s}}+\frac{p}{p^{3s}})(\frac{1}{1-p^{-2s}})\right)\\ &=\frac{\zeta(s-k)}{\zeta(2(s-k))}\prod_p\left(1+\frac{p^s+p}{(p^{s-2}+1)(p^{2s}-1)}\right)\end{aligned}$$

if $k=2$, and

$$\begin{aligned} f(s)&=\prod_p\left(1+\frac{a_3(p)b_k(p)}{p^s}+\frac{a_3(p^2)b_k(p^2))}{p^{2s}}+\cdots\right)\\ &=\prod_p\left(1+\frac{1}{p^{s-k}}+p^2\sum_{j=2}^{k}\frac{p^{k-j}}{p^{js}}+(\frac{1}{1-\frac{1}{p^{ks}}})\sum_{j=1}^{k}\frac{p^{k-j+2}}{p^{(k+j)s}}\right)\\ &=\prod_p(1+\frac{1}{p^{s-k}})\left(1+\frac{p^{s-k}}{1+p^{s-k}}\left(\sum_{j=2}^{k}\frac{p^{k-j+2}}{p^{js}}+\sum_{j=1}^{k}\frac{p^{k-j+2}}{p^{(k+j)s}-p^{js}}\right)\right)\\ &=\frac{\zeta(s-k)}{\zeta(2(s-k))}\prod_p\left(1+\sum_{j=2}^{k}\frac{p^{s-j+2}}{(p^{s-k}+1)p^{js}}+\sum_{j=1}^{k}\frac{p^{s-j+2}}{(p^{s-k}+1)(p^{(k+j)s}-p^{js})}\right)\end{aligned}$$

if $k\geq 3$.

Obviously, we have inequality

$$|a_m(n)b_k(n)|\leq n^2,\qquad \left|\sum_{n=1}^{\infty}\frac{a_m(n)b_k(n)}{n^\sigma}\right|<\frac{1}{\sigma-k-1},$$

where $\sigma>k+1$ is the real part of s. So by Perron formula [3]

$$\begin{aligned}\sum_{n\leq x}\frac{a_m(n)b_k(n)}{n^{s_0}}&=\frac{1}{2i\pi}\int_{b-iT}^{b+iT}f(s+s_0)\frac{x^s}{s}ds+O\left(\frac{x^bB(b+\sigma_0)}{T}\right)\\ &\quad+O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right)+O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right),\end{aligned}$$

where N is the nearest integer to x, $||x||=|x-N|$. Taking $s_0=0$, $b=k+2$, $T=x^{\frac{3}{2}}$, $H(x)=x^2$, $B(\sigma)=\frac{1}{\sigma-k-1}$, we have

$$\sum_{n\leq x}a_m(n)b_k(n)=\frac{1}{2i\pi}\int_{k+2-iT}^{k+2+iT}\frac{\zeta(s-k)}{\zeta(2(s-k))}R(s)\frac{x^s}{s}ds+O(x^{k+\frac{1}{2}+\varepsilon}),$$

where

$$R(s)=\begin{cases}\prod_p\left(1+\frac{p^s+p}{(p^{s-2}+1)(p^{2s}-1)}\right) & \text{if } k=2;\\ \prod_p\left(1+\sum_{j=2}^{k}\frac{p^{s-j+2}}{(p^{s-k}+1)p^{js}}+\sum_{j=1}^{k}\frac{p^{s-j+2}}{(p^{s-k}+1)(p^{(k+j)s}-p^{js})}\right) & \text{if } k\geq 3.\end{cases}$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{k+2-iT}^{k+2+iT}\frac{\zeta(s-k)}{\zeta(2(s-k))}R(s)\frac{x^s}{s}ds+O(x^{k+\frac{1}{2}+\varepsilon}),$$

we move the integral line from $s=k+2\pm iT$ to $s=k+\frac{1}{2}\pm iT$. This time, the function

$$f(s)=\frac{\zeta(s-k)x^s}{\zeta(2(s-k))s}R(s)$$

have a simple pole point at $s=k+1$ with residue $\frac{x^{k+1}}{(k+1)\zeta(2)}R(k+1)$. So we have

$$\frac{1}{2i\pi}\left(\int_{k+2-iT}^{k+2+iT}+\int_{k+2+iT}^{k+\frac{1}{2}+iT}+\int_{k+\frac{1}{2}+iT}^{k+\frac{1}{2}-iT}+\int_{k+\frac{1}{2}-iT}^{k+2-iT}\right)\frac{\zeta(s-k)x^s}{\zeta(2(s-k))s}R(s)ds$$
$$=\frac{x^{k+1}}{(k+1)\zeta(2)}R(k+1).$$

We can easy get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{k+2+iT}^{k+\frac{1}{2}+iT}+\int_{k+\frac{1}{2}-iT}^{k+2-iT}\right)\frac{\zeta(s-k)x^s}{\zeta(2(s-k))s}R(s)ds\right|$$
$$\ll\int_{k+\frac{1}{2}}^{k+2}\left|\frac{\zeta(\sigma-k+iT)}{\zeta(2(\sigma-k+iT))}R(s)\frac{x^2}{T}\right|d\sigma\ll\frac{x^{k+2}}{T}=x^{k+\frac{1}{2}};$$

and

$$\left|\frac{1}{2\pi i}\int_{k+\frac{1}{2}+iT}^{k+\frac{1}{2}-iT}\frac{\zeta(s-k)x^s}{\zeta(2(s-k))s}R(s)ds\right|\ll\int_0^T\left|\frac{\zeta(1/2+it)}{\zeta(1+2it)}\frac{x^{k+\frac{1}{2}}}{t}\right|dt\ll x^{k+\frac{1}{2}+\varepsilon}.$$

Note that $\zeta(2)=\frac{\pi^2}{6}$, from the above we have

$$\sum_{n\leq x}a_3(n)b_k(n)=\frac{6x^{k+1}}{(k+1)\pi^2}R(k+1)+O\left(x^{k+\frac{1}{2}+\varepsilon}\right).$$

This completes the proof of Theorem 1.

Let

$$f_1(s)=\sum_{n=1}^{\infty}\frac{\varphi(a_3(n)b_k(n))}{n^s},\qquad f_2(s)=\sum_{n=1}^{\infty}\frac{\sigma_\alpha(a_3(n)b_k(n))}{n^s},\qquad f_3(s)=\sum_{n=1}^{\infty}\frac{d(a_3(n)b_k(n))}{n^s}.$$

From the Euler product formula [2] and the definition of $\varphi(n), \sigma_\alpha(n)$ and $d(n)$, we also have

$$f_1(s) = \prod_p \left(1 + \frac{\varphi(a_3(p)b_k(p))}{p^s} + \frac{\varphi(a_3(p^2)b_k(p^2))}{p^{2s}} + \cdots\right)$$
$$= \prod_p \left(1 + \frac{p^2-p}{p^s} + (\frac{p^2-p}{p^{2s}} + \frac{p^3-p^2}{p^{3s}})(\frac{1}{1-p^{-2s}})\right)$$
$$= \prod_p \left(1 + \frac{1}{p^{s-2}} - \frac{1}{p^{s-1}} + \frac{(p^2-p)(p^s+p)}{p^{3s}-p^s}\right)$$
$$= \frac{\zeta(s-2)}{\zeta(2(s-2))} \prod_p \left(1 + \frac{p^{s-2}}{p^{s-2}+1}\left(\frac{(p^2-p)(p^s+p)}{p^{3s}-p^s} - \frac{1}{p^{s-1}}\right)\right)$$

if $k=2$, and

$$f_1(s) = \prod_p \left(1 + \frac{1}{p^{s-k}} - \frac{1}{p^{s-k+1}} + \sum_{j=2}^{k} \frac{p^{k-j+2}-p^{k-j+1}}{p^{js}} + \sum_{j=1}^{k} \frac{p^{k-j+2}-p^{k-j+1}}{p^{(k+j)s}-p^{js}}\right)$$
$$= \frac{\zeta(s-k)}{\zeta(2(s-k))} \prod_p \left(1 - \frac{1}{p^{s-k+1}+p} + \sum_{j=2}^{k} \frac{p^{s-j+2}-p^{s-j+1}}{(p^{s-k}+1)p^{js}} + \sum_{j=1}^{k} \frac{p^{s-j+2}-p^{s-j+1}}{(p^{s-k}+1)(p^{(k+j)s}-p^{js})}\right)$$

if $k \geq 3$.

$$f_2(s) = \frac{\zeta(s-2\alpha)}{\zeta(2(s-2\alpha))} \prod_p \left(1 + \frac{p^{s-2\alpha}}{p^{s-2\alpha}+1}\left(\frac{p^\alpha+1}{p^s} + \frac{(p^{3\alpha}-1)p^s + p^{4\alpha}-1}{(p^{3s}-p^s)(p^\alpha-1)}\right)\right)$$

if $k=2$, and

$$f_2(s) = \frac{\zeta(s-k\alpha)}{\zeta(2(s-k\alpha))} \prod_p \left(1 + \frac{p^s - p^{s-k\alpha}}{(p^{s-k\alpha}+1)(p^\alpha-1)p^s} + \sum_{j=2}^{k} \frac{p^{(3-j)\alpha+s}-p^{s-k\alpha}}{(p^{s-k\alpha}+1)(p^\alpha-1)p^{js}}\right.$$
$$\left. + \sum_{j=1}^{k} \frac{p^{(3-j)\alpha+s}-p^{s-k\alpha}}{(p^{s-k\alpha}+1)(p^\alpha-1)(p^{(k+j)s}-p^{js})}\right)$$

if $k \geq 3$.

$$f_3(s) = \frac{\zeta^3(s)}{\zeta^3(2s)} \prod_p \left(1 + \frac{p^{3s}}{(p^s+1)^3}\left(\frac{3p^s+4}{p^{3s}+p^s} - \frac{3}{p^{2s}} - \frac{1}{p^{3s}}\right)\right)$$

if $k=2$, and

$$f_3(s) = \frac{\zeta^{k+1}(s)}{\zeta^{k+1}(2s)} \prod_p \left(1 + \sum_{j=2}^{k} \frac{\left(k-j+3-\binom{k+1}{j}\right)p^{(k-j+1)s}}{(p^s+1)^{k+1}}\right.$$
$$\left. + \sum_{j=1}^{k} \frac{k-j+3}{(p^s+1)^{k+1}(p^{(j-1)s}-p^{(j-k-1)s})} - \frac{1}{(p^s+1)^{k+1}}\right)$$

if $k \geq 3$.

By Perron formula [3] and the method of proving Theorem 1, we can obtain the other results. Generally we can use the same method to study the asymptotic properties of the number sequences $a_m(n)b_k(n)$ (where $m, k \geq 2$ are fixed integers), and obtain some interesting asymptotic formulas.

REFERENCES

1. F. Smarandache, *ONLY PROBLEMS, NOT SOLUTION!*, Xiquan Publishing House, Chicago, 1993, pp. 27.
2. Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.
3. Pan Chengdong and Pan Chengbiao, *Foundation of Analytic Number Theory*, Science Press, Beijing, 1997, pp. 98.