On a dual of the Pseudo-Smarandache function

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1 Introduction

In paper [3] we have defined certain generalizations and extensions of the Smarandache function. Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an arithmetic function with the following property: for each $n \in \mathbb{N}^*$ there exists at least a $k \in \mathbb{N}^*$ such that $n|f(k)$. Let

$$F_f : \mathbb{N}^* \to \mathbb{N}^* \text{ defined by } F_f(n) = \min\{k \in \mathbb{N}^* : \ n|f(k)\}. \tag{1}$$

This function generalizes many particular functions. For $f(k) = k!$ one gets the Smarandache function, while for $f(k) = \frac{k(k+1)}{2}$ one has the Pseudo-Smarandache function Z (see [1], [4-5]). In the above paper [3] we have defined also dual arithmetic functions as follows: Let $g : \mathbb{N}^* \to \mathbb{N}^*$ be a function having the property that for each $n \geq 1$ there exists at least a $k \geq 1$ such that $g(k)|n$.

Let

$$G_g(n) = \max\{k \in \mathbb{N}^* : \ g(k)|n\}. \tag{2}$$

For $g(k) = k!$ we obtain a dual of the Smarandache function. This particular function, denoted by us as S_* has been studied in the above paper. By putting $g(k) = \frac{k(k+1)}{2}$ one obtains a dual of the Pseudo-Smarandache function. Let us denote this function, by analogy by Z_*. Our aim is to study certain elementary properties of this arithmetic function.

2 The dual of yhe Pseudo-Smarandache function

Let

$$Z_*(n) = \max\left\{m \in \mathbb{N}^* : \frac{m(m+1)}{2} | n\right\}. \tag{3}$$

Recall that

$$Z(n) = \min\left\{k \in \mathbb{N}^* : n | \frac{k(k+1)}{2}\right\}. \tag{4}$$

First remark that

$$Z_*(1) = 1 \quad \text{and} \quad Z_*(p) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{5}$$

where p is an arbitrary prime. Indeed, $\frac{2 \cdot 3}{2} = 3|3$ but $\frac{m(m+1)}{2}|p$ for $p \neq 3$ is possible only for $m = 1$. More generally, let $s \geq 1$ be an integer, and p a prime. Then:

Proposition 1.

$$Z_*(p^s) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \tag{6}$$

Proof. Let $\frac{m(m+1)}{2}|p^s$. If $m = 2M$ then $M(2M+1)|p^s$ is impossible for $M > 1$ since M and $2M+1$ are relatively prime. For $M = 1$ one has $m = 2$ and $3|p^s$ only if $p = 3$. For $m = 2M-1$ we get $(2M-1)M|p^k$, where for $M > 1$ we have $(M, 2M-1) = 1$ as above, while for $M = 1$ we have $m = 1$.

The function Z_* can take large values too, since remark that for e.g. $n \equiv 0 (\mathrm{mod} 6)$ we have $\frac{3 \cdot 4}{2} = 6|n$, so $Z_*(n) \geq 3$. More generally, let a be a given positive integer and n selected such that $n \equiv 0(\mathrm{mod}\, a(2a+1))$. Then

$$Z_*(n) \geq 2a. \tag{7}$$

Indeed, $\frac{2a(2a+1)}{2} = a(2a+1)|n$ implies $Z_*(n) \geq 2a$.

A similar situation is in

Proposition 2. Let q be a prime such that $p = 2q - 1$ is a prime, too. Then

$$Z_*(pq) = p. \tag{8}$$

Proof. $\frac{p(p+1)}{2} = pq$ so clearly $Z_*(pq) = p$.

Remark. Examples are $Z_*(5\cdot 3) = 5$, $Z_*(13\cdot 7) = 13$, etc. It is a difficult open problem that for infinitely many q, the number p is prime, too (see e.g. [2]).

Proposition 3. For all $n \geq 1$ one has

$$1 \leq Z_*(n) \leq Z(n). \tag{9}$$

Proof. By (3) and (4) we can write $\frac{m(m+1)}{2}|n|\frac{k(k+1)}{2}$, therefore $m(m+1)|k(k+1)$. If $m > k$ then clearly $m(m+1) > k(k+1)$, a contradiction.

Corollary. One has the following limits:

$$\underline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 0, \quad \overline{\lim}_{n\to\infty} \frac{Z_*(n)}{Z(n)} = 1. \tag{10}$$

Proof. Put $n = p$ (prime) in the first relation. The first result follows by (6) for $s = 1$ and the well-known fact that $Z(p) = p$. Then put $n = \frac{a(a+1)}{2}$, when $\frac{Z_*(n)}{Z(n)} = 1$ and let $a \to \infty$.

As we have seen,

$$Z\left(\frac{a(a+1)}{2}\right) = Z_*\left(\frac{a(a+1)}{2}\right) = a.$$

Indeed, $\frac{a(a+1)}{2}\Big|\frac{k(k+1)}{2}$ is true for $k = a$ and is not true for any $k < a$. In the same manner, $\frac{m(m+1)}{2}\Big|\frac{a(a+1)}{2}$ is valied for $m = a$ but not for any $m > a$. The following problem arises: What are the solutions of the equation $Z(n) = Z_*(n)$?

Proposition 4. All solutions of equation $Z(n) = Z_*(n)$ can be written in the form $n = \frac{r(r+1)}{2}$ $(r \in \mathbb{N}^*)$.

Proof. Let $Z_*(n) = Z(n) = t$. Then $n|\frac{t(t+1)}{2}|n$ so $\frac{t(t+1)}{2} = n$. This gives $t^2 + t - 2n = 0$ or $(2t+1)^2 = 8n+1$, implying $t = \frac{\sqrt{8n+1}-1}{2}$, where $8n+1 = m^2$. Here m must be odd, let $m = 2r+1$, so $n = \frac{(m-1)(m+1)}{8}$ and $t = \frac{m-1}{2}$. Then $m - 1 = 2r$, $m + 1 = 2(r+1)$ and $n = \frac{r(r+1)}{2}$.

Proposition 5. One has the following limits:

$$\lim_{n\to\infty} \sqrt[n]{Z_*(n)} = \lim_{n\to\infty} \sqrt[n]{Z(n)} = 1. \tag{11}$$

Proof. It is known that $Z(n) \leq 2n-1$ with equality only for $n = 2^k$ (see e.g. [5]). Therefore, from (9) we have

$$1 \leq \sqrt[n]{Z_*(n)} \leq \sqrt[n]{Z(n)} \leq \sqrt[n]{2n-1},$$

and by taking $n \to \infty$ since $\sqrt[n]{2n-1} \to 1$, the above simple result follows.

As we have seen in (9), upper bounds for $Z(n)$ give also upper bounds for $Z_*(n)$. E.g. for $n =$ odd, since $Z(n) \leq n-1$, we get also $Z_*(n) \leq n-1$. However, this upper bound is too large. The optimal one is given by:

Proposition 6.

$$Z_*(n) \leq \frac{\sqrt{8n+1}-1}{2} \text{ for all } n. \tag{12}$$

Proof. The definition (3) implies with $Z_*(n) = m$ that $\frac{m(m+1)}{2} | n$, so $\frac{m(m+1)}{2} \leq n$, i.e. $m^2 + m - 2n \leq 0$. Resolving this inequality in the unknown m, easily follows (12). Inequality (12) cannot be improved since for $n = \frac{p(p+1)}{2}$ (thus for infinitely many n) we have equality. Indeed,

$$\left(\sqrt{\frac{8(p+1)p}{2}+1}-1\right)/2 = \left(\sqrt{4p(p+1)+1}-1\right)/2 = [(2p+1)-1]/2 = p.$$

Corollary.

$$\underline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = 0, \quad \overline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} = \sqrt{2}. \tag{13}$$

Proof. While the first limit is trivial (e.g. for $n =$ prime), the second one is a consequence of (12). Indeed, (12) implies $Z_*(n)/\sqrt{n} \leq \sqrt{2}\left(\sqrt{1+\frac{1}{8n}} - \sqrt{\frac{1}{8n}}\right)$, i.e. $\overline{\lim}_{n\to\infty} \frac{Z_*(n)}{\sqrt{n}} \leq \sqrt{2}$. But this upper limit is exact for $n = \frac{p(p+1)}{2}$ $(p \to \infty)$.

Similar and other relations on the functions S and Z can be found in [4-5].

An inequality connecting $S_*(ab)$ with $S_*(a)$ and $S_*(b)$ appears in [3]. A similar result holds for the functions Z and Z_*.

Proposition 7. For all $a, b \geq 1$ one has

$$Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}, \tag{14}$$

$$Z(ab) \geq \max\{Z(a), Z(b)\} \geq \max\{Z_*(a), Z_*(b)\}. \tag{15}$$

Proof. If $m = Z_*(a)$, then $\frac{m(m+1)}{2} | a$. Since $a|ab$ for all $b \geq 1$, clearly $\frac{m(m+1)}{2} | ab$, implying $Z_*(ab) \geq m = Z_*(a)$. In the same manner, $Z_*(ab) \geq Z_*(b)$, giving (14).

Let now $k = Z(ab)$. Then, by (4) we can write $ab | \frac{k(k+1)}{2}$. By $a|ab$ it results $a | \frac{k(k+1)}{2}$, implying $Z(a) \leq k = Z(ab)$. Analogously, $Z(b) \leq Z(ab)$, which via (9) gives (15).

Corollary. $Z_*(3^s \cdot p) \geq 2$ for any integer $s \geq 1$ and any prime p. (16)

Indeed, by (14), $Z_*(3^s \cdot p) \geq \max\{Z_*(3^s), Z(p)\} = \max\{2, 1\} = 2$, by (6).

We now consider two irrational series.

Proposition 8. The series $\sum_{n=1}^{\infty} \frac{Z_*(n)}{n!}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} Z_*(n)}{n!}$ are irrational.

Proof. For the first series we apply the following irrationality criterion ([6]). Let (v_n) be a sequence of nonnegative integers such that

(i) $v_n < n$ for all large n;

(ii) $v_n < n - 1$ for infinitely many n;

(iii) $v_n > 0$ for infinitely many n.

Then $\sum_{n=1}^{\infty} \frac{v_n}{n!}$ is irrational.

Let $v_n = Z_*(n)$. Then, by (12) $Z_*(n) < n - 1$ follows from $\frac{\sqrt{8n+1} - 1}{2} < n - 1$, i.e. (after some elementary fact, which we omit here) $n > 3$. Since $Z_*(n) \geq 1$, conditions (i)-(iii) are trivially satisfied.

For the second series we will apply a criterion from [7]:

Let $(a_k), (b_k)$ be sequences of positive integers such that

(i) $k | a_1 a_2 \ldots a_k$;

(ii) $\frac{b_{k+1}}{a_{k+1}} < b_k < a_k$ $(k \geq k_0)$. Then $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{b_k}{a_1 a_2 \ldots a_k}$ is irrational.

Let $a_k = k$, $b_k = Z_*(k)$. Then (i) is trivial, while (ii) is $\frac{Z_*(k+1)}{k+1} < Z_*(k) < k$. Here $Z_*(k) < k$ for $k \geq 2$. Further $Z_*(k+1) < (k+1) Z_*(k)$ follows by $1 \leq Z_*(k)$ and $Z_*(k+1) < k+1$.

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