# To Enjoi is a Permanent Component of Mathematics

by
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## 1. The Theorem of Platon

Studying the properties of the proportions the peoples of the antiquity could build using the ruler and the compasses. For example if instead of a square of side $a$ it was required the construction of another square, of side $x$ determined by the condition that the new square has a double area, so

$$x^2 = 2a^2 \tag{1}$$

Pithagora's descendents used to write this relation as

$$\frac{a}{x} = \frac{x}{2a} \tag{2}$$

and used to build an isosceles rectangular triangle having its hypotenuse $2a$.

The celebrated philosopher of the antiquity *Platon* (427 - 347 B.C.) was greatly interested in Mathematics, especialy in connections with the so called "*solid numbers*", that is numbers of the form

$$a \cdot b \cdot c$$

representing a volume.

This sympathy is also due to a famous event even today.

In the Greek city Athens there was an epidemic diseare that killed many peoples. The inhabitants anced the oracle of Delphi (a town in Delos, the smallest of the Ciclade isles) what to do in order to save themselves.

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The gods asked the prierts of the temple to replace their cubic altar with a new one having a double volume.

The prierts appealed to the greatest mathematicians of the time to get the solution.

The problem is to calculate the lenght $x$ of the side of a cube such that

$$x^3 = 2a^3$$

That is

$$x = 2^{\frac{1}{3}} a \tag{3}$$

But the peoples of those times didn't know any method to calculate, not even approximatively, the radicals over to two. Only in the fifth century A.D. the Indians used the approximation in order to extract the cubic root:

$$(a^3 + b)^{\frac{1}{3}} \simeq a + \frac{b}{3a^2}$$

where $a^3$ is the greatest perfect cube not exceding the number $a^3 + b$.

The problem (3) can't be solved using only the rule and the compasses.

Let us observe that this problem is a particular problem on *solid numbers*, and of course it is unsolvable by of *only one* proportion of kind (2).

However Platon observed that this problem could be solved using *two proportions*. Namely, he affirmed that:

<u>**Theorem of Platon.**</u> *While one simple proportion is enought to connect two plane numbers (numbers of the form $a \cdot b$), three proportions are necessary to connect twoo solide numbers.*

The solution of the problem of Delos is then obtained by Platon approximatively writing

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a} \tag{4}$$

Indeed, from (4) we obtain

$$x^2 ay \quad \text{and} \quad y^2 = 2ax \tag{5}$$

so $x^3 = 2a^3$.

Platon and others [Archytas of Tarent ($\sim$ 380 B.C.), Eudoxus (408 - 355 B.C.), Appollonios of Perga (260 - 170 B.C.)] imaginated approximate solutions of the equation (4), rather difficult, which, of course, could be simplified in the course of time.

Today, we can easily find an approximate solution to the system (5) through drawning the two parabolas or intersecting one of these parabolas with the circle

$$x^2 + y^2 - 2ax - ay = 0$$

obtained through adding the equations of the two parabolas.

## 2. A method to construct convergent sequences

The name of *Leonard Euler* (1707 - 1783) is known amoung the young people loving mathematics, especially because of the sequence given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n \tag{6}$$

It is said that this sequence is monotonous and bounded, converging to a constant $\gamma \in (0, 1)$, known as *Euler's constant.*

This constant appears in many occasions in mathematics. For instance if $d(n)$ is the number of (positive) divisors of the positive integer $n$, then it is proved that

$$\frac{1}{n} \sum_{i=1}^{n} d(i) \simeq \ln n + 2\gamma - 1$$

Considering the sequence (6) and proving his convergence Euler has etablished a connection between the following two sequences

$$b_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \quad \text{and} \quad c_n = \ln n$$

both converging to infinity.

To prove the monotonicity and boundness of the sequence $(a_n)_{n \in N^*}$ it is used a well known theorem, does to the *count Luis de Lagrange* (1736 - 1813). This method may be generalised in the following way:

**Proposition.** *Let $f : [1, \infty) \longrightarrow R$ a derivable function with the property that $f$ and $f'$ are monotonous, but of different monotonicity (that is either $f$ increase and $f'$ decrease or $f$ decrease and $f'$ increase).*

*Then the sequence*

$$x_n = f'(1) + f'(2) + ... + f'(n) - f(n) \tag{7}$$

*is convergent.*

*Proof.* The proof is analogous with that of Euler's sequence (6).

Indeed, let us suppose that $f$ is increasing and $f'$ is decreasing. For the monotoncity of the sequence $(x_n)_{n \in N^8}$ we obtains:

$$x_{n+1} - x_n = f'(n+1) - (f(n+1) - f(n))$$

and applying the theorem of Lagrange to the function $f$ on the interval $[k, k+1]$ it results:

$$(\exists) \ c_k \in (k, k+1) \quad \text{such that} \quad f(k+1) - f(k) = f'(c_k) \tag{8}$$

and

$$k < c_k < k+1 \Longrightarrow f'(k) > f'(c_k) > f'(k+1) \tag{9}$$

so

$$x_{n+1} - x_n = f'(n+1) - f'(c_n) < 0$$

because $f'$ is decreasing.

We have now to find a lower bound of the sequence (7). For this we write the implication (9) for every $k = 1, 2, ...,$and we get:

$$\begin{aligned} &1 < c_1 < 2 \Longrightarrow f'(1 > f'(c_1) > f'(2) \\ &2 < c_2 < 3 \Longrightarrow f'(2) > f'(c_2) > f'(3) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &n < c_n < n+1 \Longrightarrow f'(n) > f'(c_n) > f'(n+1) \end{aligned} \tag{10}$$

So,

$$\begin{aligned} x_n = f'(1) + f'(2) + ... + f'(n) - f(n) > \\ > f'(c_1) + f'(c_2) + ... + f'(c_n) - f(n) \end{aligned}$$

Writing now the equalities (8) for $k = 1, 2, ...n$ and adding, it results:

$$f'(c_1) + f'(c_2) + \ldots + f'(c_n) = f(n+1) - f(1)$$

so $x_n \geq f(n+1) - f(1) - f(n) > -f(1)$ because $f$ is increasing.

Of course, the limit point of this sequence is between $-f(1)$ and $x_1 = f'(1) - f(1)$.

This proposition permet to construct many convergent sequences of the form (7)

Indeed,

1) considering the increasing function $f(x) = 2\sqrt{x}$, whose derivative $f'(x) = 1/\sqrt{x}$ is decreasing, it results that the sequence

$$x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

has a limit point $l \in [-2, -1]$.

2) considering the function $f(x) = \ln(\ln x)$ it results that the sequence

$$x_n = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \ldots + \frac{1}{n\ln n} - \ln(\ln n)$$

is convergent to a point $l \in [-\ln(\ln 2), \frac{1}{2\ln 2} - \ln(\ln 2)]$.

3) the sequence

$$x_n = 2\left(\frac{\ln 2}{2} + \frac{\ln 3}{3} + \ldots + \frac{\ln n}{n}\right) - \ln^2 n$$

as well as

$$x_n = \frac{\ln^k 2}{2} + \frac{\ln^k 3}{3} + \ldots + \frac{\ln^k n}{n} - \frac{\ln^{k+1} n}{k+1}$$

are convergent sequences, and, of course, the reader may construct himself many other convergent sequences, using the same method.

It is interesting to mentione that by means of the same way as in the proof of the above proposition it may be proved the following curious inequalities:

$$1998 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{10^6}} < 1999$$

and, more general,

$$2 \cdot 10^k - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{10^{2k}}} < 2 \cdot 10^k - 1$$

or these

$$\frac{p}{p-1}(a^{k(p-1)} - 1) < 1 + \frac{1}{2^{1/p}} + \frac{1}{3^{1/p}} + ... + \frac{1}{(a^{pk})^{1/p}} < \frac{p}{p-1}(a^{k(p-1)} - \frac{1}{p})$$

## 3. The Problem of Titeica

The Romanian mathematician *Gh. Titeica* (1873 - 1939) while in a waiting room and because time hardly passed, started drawing circles on a newspaper margin, using a coin.

In this playing with it, he begun to move the coin so that it have a fixed point on the circumference of a cercle. Because he had to wait for a long while, he had the time to find out that drawing three circles in which the coin had a fixed point on the circumference, the circles intersected two by two in three points (called $A$, $B$, and $C$) over which the coin was exactly superposed.

Of course, the three points $A$, $B$, and $C$ make a circle. The novelty was that this circle seemed to have the same radius as the circles drawn with the coin.

When he reached home, Titeica proved that indeed:

**The Problem of Titeica.** *If three circles of the same radius $r$ have a common fixed point $M$, they still intersect two by two in the points $A, B, C$ which make another circle with the same radius $r$.*

*Proof.* Because we have $MC_1 = MC_2 = MC_3$ (see figure below) it results that $M$ is the centre of the circumscribed circle of the triangle determined by the points $C_1, C_2, C_3$.

Now, it is sufficient to prove the equality (congruence) between this triangle and the triangle determined by the points $A, B, C$.

We have:

$$AB \equiv C_2C_3 \text{ ( because } \Delta AC_1B \equiv \Delta C_2CC_3)$$
$$AC \equiv C_1C_3 \text{ (because } \Delta AC_2C \equiv \Delta C_1BC_3)$$
$$BC \equiv C_1C_2 \text{ (because } \Delta BC_3C \equiv \Delta C_1AC_2)$$

and the theorem is proved.

## 4. Hexagons in Pascal's Triangle

The hexagon $AC_2CC_3BC_1$ used in the proof of the problem of *Titeica* is in connection with some cercles. Now we shall make in evidence other hexagons. this time lied with a triangle, the celebrate triangle of *Pascal.*

In 1654 *Blaise Pascal* (1623 - 1662) published the paper "*On an Arithmetical Triangle*" in which studied the properties of the numbers in the triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & 2 & 1 & & & & \\ & & & 1 & 3 & & 3 & 1 & & & \\ & & 1 & 4 & & 6 & & 4 & & 1 & \\ & 1 & 5 & & 10 & & 10 & & 5 & & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

constructed such that the $n-row$ contains the elements

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{k-1}, \binom{n}{k}, \binom{n}{k+1}, \ldots, \binom{n}{n}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In the sequel we shall focus the attention on the following elements in this triangle:

$$\begin{array}{ccccc} & \binom{n-1}{k-1} & & \binom{n-1}{k} & \\ \binom{n}{k-1} & & \binom{n}{k} & & \binom{n}{k+1} \\ & \binom{n+1}{k} & & \binom{n+1}{k+1} & \end{array}$$

For simplicity we note

$$A = \binom{n-1}{k-1}, B = \binom{n}{k-1}, C = \binom{n+1}{k}, D = \binom{n+1}{k+1}$$
$$E = \binom{n}{k+1}, F = \binom{n-1}{k}, \quad \text{and} \quad X = \binom{n}{k}$$

so it results the configuration

$$\begin{array}{ccccc} & A & & F & \\ B & & X & & E \\ & C & & D & \end{array}$$

The multiplicative equality

$$A \cdot C \cdot E = B \cdot D \cdot F \tag{11}$$

was found by *V. E. Hoggatt Jr.* and *W. Hansell* [5]. Therefore this configuration is called *"Hoggat-Hansell perfect square hexagon"*.

This hexagon has also the following interesting property, found in [2]:

$$g.c.d.(A, C, E) = g.c.d.(B, D, F) \tag{12}$$

where *g.c.d.* is the abreviation for the greatest common divisor.

The identities (11) and (12) are the first two non-trivial examples of translatable identities of binomial coefficients and are called *"the Star of David theorem"*.

The lower common multiple (*l.c.m.*) counterpart of the identity (12), namely

$$l.c.m.(A, C, E) = l.c.m.(B, D, F) \tag{13}$$

does not hold on Pascal's triangle and it has been a long-standing open question wheter there exists any mathematically non-trivial and/or artistically interesting configurations which give a *translatable l.c.m. identity* of type (12).

*S. Ando and D. Sato* have proved [2] that the answer to this question is *"yes"*. They have proved that:

**Theorem.** *(Pisa triple equality theorem) There exists a configuration which gives simultaneously equal product, equal g.c.d. and equal l.c.m. properties on binomial, Fibonacci-binomial and their modified coefficients.*

A *Fibonacci-binomial coefficient* (or *Fibonomial-coefficient*) is the number defined by:

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{F_1 \cdot F_2 \ldots F_n}{F_1 \cdot F_2 \ldots F_k \cdot F_1 \cdot F_2 \ldots F_{n-k}}$$

where $F_i$ is the $i-th$ *Fibonacci number*, i.e.

$$F_1 = F_2 = 1, \quad F_{n+2} = F_n + F_{n+1}, \text{ for } n = 1, 2, \ldots$$

All Fibonomial coefficients are positive integers. and the triangular array of these numbers has a structure similar to Pascal's triangle.

A. P. Hilmann and V. E. Hoggatt Jr. investigated the similarities with Pascal's triangle and showed that the original Star of David theorem also holds on this Fibonacci version of the Pascal-like triangle.

The modified binomial coefficient is defined as

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{(n+1)!}{k!(n-k)!} = (n+1) \left( \begin{matrix} n \\ k \end{matrix} \right)$$

It is proved that the translatable product and *l.c.m.* equalities, similar to (11) and (13), but not the *g.c.d.* equality (12), hold for the array of modified binomial coefficients.

The two Pascal like number array can be combined further to define the *modified Fibonacci coefficient*, given by:

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \frac{F_1 \cdot F_2 \ldots F_{n+1}}{F_1 \cdot F_2 \ldots F_k \cdot F_1 \cdot F_2 \ldots F_{n-k}} = F_{n+1} \left[ \begin{matrix} n \\ k \end{matrix} \right]$$

S. Ando and D. Sato announced at the *third International Conference on Fibonacci Numbers and their Applications* (held in Pisa, Italy, July 25 -29, 1988) some interesting results concerning *g.c.d.* and *l.c.m.* properties on configurations like these reproduced below. We mention here only the following:

<u>**Theorem**</u> (Sakasa - Fuji quadruple equality theorem). *The configuration of Fujiyama* (see below) *has equal g.c.d. and equal l.c.m. properties on Fibonacci - Pascal's triangle,while its upside down configuration* (called *SAKASA - FUJI*, in japonese) *has equal g.c.d. and equal l.c.m. properties on modifies Pascal's and modified Fibonacci - Pascal's triangle.*

<u>**Theorem**</u> (Universal equality theorem). *The Julia's snowflake and its upside down configuration both give translatable simultaneously equal product* (symbolised below by the Greek letter $\Pi$), *equal g.c.d. and equal l.c.m. properties on Pascal's triangle, Fibonacci - Pascal's triangle and modified Fibonacci - Pascal's triangle*

We reproduce here, after [2] these configurations.

S. Ando and D. Sato in their paper explained with amability the terminology used for these configurations.

Thus one of the configurations is named in memoriam of Professor *Julia Robinson* for the friendship and support given to the authors during many years of mathematical associations.

*Fujiyama* is a highly symetric triangular mountain near Tokio, and *Saskatchewan* is a name of a province in western Canada, where the first non - trivial mutualy exclusive equal *g.c.d.* $-$ *l.c.m.* configuration was constructed.

## 5. The Smarandache Function

This function is originated from the exiled Romanian Professor *Florentin Smarandache* and it is defined as follows:

*For any non – null n, S(n) is the smallest integer such that S(n)! is divisible byn.*

To calculate the value of $S(n)$, for a given $n$, we need to use twoo numerical scale, as we shall see in the following.

**A strange addition.** A (standard) numerical scale is a sequence

$$(h)\ :\ 1,\ a_1,\ a_2, ..., a_i, ... \tag{14}$$

where $a_i = h^i$, for a fixed $h > 1$.

By means of such a sequence every integer $n \in N$ may be writen as

$$n_{(h)} = \varphi_k a_k + \varphi_{k-1} a_{k-1} + ... + \varphi_0$$

and we can use the notation

$$n_{(h)} = \overline{\varphi_k \varphi_{k-1} ... \varphi_0}$$

The integers $\varphi_i$ are called *"digits"* and verifie the inequalities

$$0 \leq \varphi_i \leq h - 1$$

For the scale given by the sequence (14) it is trouth the recurence relation

$$a_{i+1} = h \cdot a_i \tag{15}$$

which permet numerical calculus, as additions, substractions, etc.

The standard scale (14) was been generalised, considering *an arbitrary increassing sequence:*

$$(g) \ : \ 1, \ b_1, \ b_2, ..., b_i, ...$$

and knowing a corresponding recurence relation.

For instance the Fibonacci sequence:

$$F_1 = 1, \ F_2 = 2, \ \text{ and } F_{i+1} = F_i + F_{i-1}$$

is such a generalised scale, for which the digits are only the integers 0 and 1.

Another generalised numerical scale is the scale defined by the sequence:

$$[p] \ : \ 1, \ b_1, \ b_2, ..., b_i, ... \tag{16}$$

with

$$b_i = \frac{p^i - 1}{p - 1} \tag{17}$$

and $p$ a prime number.

This scale verifies the recurence

$$b_{i+1} = p \cdot b_i + 1 \tag{18}$$

and is used in the calculus of Smarandache function.

Let us observe that because of the diference between the recurences (15) and (18) we have essentially different ruls for the calculus in the scale $[p]$. To ilustrate these differences let we consider the generalised scale [5]:

$$[5] \ : \ 1, \ 6, \ 31, \ 156, \ ...$$

and the integer $m = 150_{(10)}$, which becomes $m_{[5]} = 442$, in the scale [5]. Indeed, because

$$a_i(5) \leq 150 \Longleftrightarrow (p^i - 1)/(p - 1) \leq 150 \Longleftrightarrow p^i \leq 150(p - 1) + 1 \Longleftrightarrow$$
$$\Longleftrightarrow i \leq \log_5(150(p - 1) + 1)$$

it results that the greatest $a_i(5)$ for which $a_i(5) \leq 150$ is $a_3(5) = 31$. Then the first digit of the number $m_{[5]}$ is

$$k_3 = [\frac{150}{a_3(5)}] = 4$$

so, $150 = 4a_3(5) + 26$.

For $m_1 = 26$ it results that the greatest $a_i(5)$ for which $a_i(5) \leq 26$ is $a_2(5) = 6$ and the corresponding digit is:

$$k_2 = [\frac{26}{6}] = 4$$

so, $150 = 4a_3(5) + 4a_2(5) + 2 = 442_{[5]}$.

If we consider in addition the numbers:

$$n_{[5]} = 412, \quad r_{[5]} = 44$$

then

$$\begin{array}{r} m + n + r = 442+ \\ 412 \\ 44 \\ dcba \end{array}$$

From the recurence (18) it results that we need to start the addition from the column corresponding to $a_2(5)$:

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5)$$

Now, using an unit from the first collumn it results:

$$5a_2(5) + 4a_2(5) = a_3(5) + 4a_2(5), \quad \text{so } b = 4$$

Continuing, $4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$ and using a new unit from the first collumn it results

$$4a_3(5) + 4a_3(5) + a_3(5) = a_4(5) + 4a_3(5), \quad \text{so } c = 4 \text{ and } d = 1$$

Finally, adding the remainder units, $4a_1(5)+2a_1(5) = 5a_1(5)+a_1(5) = 5a_1(5)+1 = a_2(5)$, it results that $b$ must be modified and $a = 0$. So, $m + n + r = 1450_{[5]}$.

An other particularity for the calculus in the scale $[p]$ results from the fact that *in this scale the last non-zero digit may be even p*. This particularity is a consequence of the recurence relation (18).

**Which are the numbers with the factorial ending in 1,000 zeros?** The answer to this question is in a strong connection with the Smarandache function.

For this reason let us observe first that if

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots . p_t^{\alpha_t} \tag{19}$$

is the decomposition of a given positive integer $n$ into primes, then as an imediate consequence of the definition of $S$ it results

$$S(n) = \max_{i=1,t} (S(p_i^{\alpha_i})) \tag{20}$$

Now, for $n = 10^{1,000}$ it results that $S(n)!$ is a multiple of $10^{1,000}$ and it is the smallest positive integer with this property.

We have

$$S(10^{1,000}) = S(2^{1,000} \cdot 5^{1,000}) = \max(S(2^{1,000}), S(5^{1,000})) = S(5^{1,000})$$

Indeed, for the calculus of $S(p^\alpha)$ we can use the formula:

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)}$$

which signify that the value of the function $S$ for $p^\alpha$ is obtained multiplying by $p$ the number obtained *writting* the exponent $\alpha$ in the generalised scale $[p]$ and *reading* it in the scale $(p)$.

So, we have:

$$S(2^{1,000}) = 2((1,000)_{[2]})_{(2)} = 2((111111100)_{[2]})_{(2)} = 508$$
$$S(5^{1,000}) = 5(11201_{[5]})_{(5)} = 4005$$

and it results that $n = 4005$ is the smallest positive integer who's factorial end in 1,000 zeros.

The next integers with this property are 4006, 4007, 4008, and 4009, because the factorial of 4010 has 1,001 zeros.

**Smarandache magic square.** For $n \geq 2$ let $A$ be a set of $n^2$ elements and $l$ a $n-$arry law defined on $A$. The *Smarandache magic square of order* $n$ is a 2 square array of rows of elements of $A$ arranged so that the law $l$ applied to each horizontal and vertical row and diagonal give the same result.

*Mike R. Mudge*, considering such squares, poses the following questions (see Smarandache Function Journal, Vol. 7, No. 1, 1996):

1) Can you find such magic square of order at least 3 or 4, when $A$ is a set of prime numbers and $l$ the addition?

2) Same question when $A$ is a set of square numbers, or cube numbers, or special numbers. For example Fibonacci or Lucas numbers, triangular numbers, Smarandache quotients (i.e. $q(m)$ is the smallest $k$ such that $mk$ is a factorial).

An interesting law may be

$$l(a_1, a_2, ..., a_n) = a_1 + a_2 + -a_3 + a_4 - a_5 + ...$$

Now some examples of *Smarandache Magic Square:*

If $A$ is a set of *prime numbers* and $l$ is the operation of addition such magic squares, with the constant in brackets, are:

$$\begin{vmatrix} 83 & 89 & 41 \\ 29 & 71 & 113 \\ 101 & 53 & 59 \\ & (213) & \end{vmatrix}, \begin{vmatrix} 101 & 491 & 251 \\ 431 & 281 & 131 \\ 311 & 71 & 461 \\ & (843) & \end{vmatrix}, \begin{vmatrix} 71 & 461 & 311 \\ 521 & 281 & 41 \\ 251 & 101 & 491 \\ & (843) & \end{vmatrix}$$

$$\begin{vmatrix} 113 & 149 & 257 \\ 317 & 173 & 29 \\ 89 & 197 & 233 \\ & (519) & \end{vmatrix}, \begin{vmatrix} 97 & 907 & 557 & 397 & 197 \\ 367 & 167 & 67 & 877 & 677 \\ 997 & 647 & 337 & 137 & 37 \\ 107 & 157 & 967 & 617 & 307 \\ 587 & 277 & 227 & 127 & 937 \\ & & (2155) & & \end{vmatrix}$$

The *multiplication* magic square

$$\begin{vmatrix} 18 & 1 & 12 \\ 4 & 6 & 9 \\ 3 & 36 & 2 \\ & (216) & \end{vmatrix}$$

is such that the constant 216 may be obtained by multiplication of the elements in any *row/column/principal diagonal.*

A geometric magic square is obtained using elements which are a given base raised to the powers of the corresponding elements of a magic square it is clearly a multiplication magic square.

For instance, considering

$$\begin{vmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \\ & (15) & \end{vmatrix} \text{ and base 2 it results } \begin{vmatrix} 256 & 2 & 64 \\ 8 & 32 & 128 \\ 16 & 512 & 4 \\ & (2^{15}) & \end{vmatrix}$$

*Talisman Magic Squares* are a relatively new concept, contain the integers from 1 to $n^2$ in such a way that the difference between any integer and its neighbours (either *row-*, *column-*, or *diagonal-wise*) is greather than some given constant:

$$\begin{vmatrix} 5 & 15 & 9 & 12 \\ 10 & 1 & 6 & 3 \\ 13 & 16 & 11 & 14 \\ 2 & 8 & 4 & 7 \\ & & (2) & \end{vmatrix}$$

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