ERDÖS-SMARANDACHE NUMBERS

Sabin Tabirca* Tatiana Tabirca**

*Transilvania University of Brasov, Computer Science Department

**University of Manchester, Computer Science Department

The starting point of this article is represented by a recent work of Finch [2000]. Based on two asymptotic results concerning the Erdos function, he proposed some interesting equations concerning the moments of the Smarandache function. The aim of this note is give a bit modified proof and to show some computation results for one of the Finch equation. We will call the numbers obtained from computation '*the Erdos-Smarandache Numbers*'. The Erdos-Smarandache number of order 1 is obtained to be the Golomb-Dickman constant.

1. INTRODUCTION

We briefly present the results used in this article. These concern the relationship between the Smarandache and the Erdos functions and some asymptotic equations concerning them. The Smarandache and Erdos functions are important functions in Number Theory defined as follows:

- The Smarandache function [Smarandache, 1980] is $S: N^* \to N$,

$$S(n) = \min\{k \in N | k! X n\} \; (\forall n \in N^*). \tag{1}$$

- The Erdos function is $P: N^* \to N$,

$$P(n) = \min\{p \in N \mid n X p \wedge p \text{ is prim}\} (\forall n \in N^* \setminus \{1\}),\; P(1) = 0. \tag{2}$$

Their main properties are:

$$(\forall a,b \in N^*)\; (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\},\; P(a \cdot b) = \max\{P(a), P(b)\}. \tag{3}$$

$$(\forall a \in N^*)\; P(a) \le S(a) \le a \text{ and the equalities occur } \textit{iif} \text{ a is prim.} \tag{4}$$

An important equation between these functions was found by Erdos [1991]

$$\lim_{n \to \infty} \frac{\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right|}{n} = 0, \tag{5}$$

which was extended by Ford [1999] to

$$\left|\left\{i = \overline{1,n} \mid P(i) < S(i)\right\}\right| = n \cdot e^{-\left(\sqrt{2} + a_n\right)\sqrt{\ln n \cdot \ln \ln n}}, \text{ where } \lim_{n \to \infty} a_n = 0. \tag{6}$$

Equations (5-6) are very important because create a similarity between these functions especially for asymptotic properties. Moreover, these equations allow us to translate convergence properties of the Smarandache function to convergence properties on the Erdos function and vice versa. The main important equations that have been obtained using this translation are presented in the following.

The average values

$\frac{1}{n}\sum_{i=2}^{n} S(i) = O(\frac{n}{\log n})$ [Luca, 1999], $\frac{1}{n}\sum_{i=2}^{n} P(i) = O(\frac{n}{\log n})$ [Tabirca, 1999a]

and their generalizations

$$\frac{1}{n}\sum_{i=2}^{n} P^{a}(i) = \frac{\zeta(a+1)}{a+1}\cdot\frac{n^{a}}{\ln(n)} + O\left(\frac{n^{a}}{\ln^{2}(n)}\right) \text{ [Knuth and Pardo 1976]}$$

$$\frac{1}{n}\sum_{i=2}^{n} S^{a}(i) = \frac{\zeta(a+1)}{a+1}\cdot\frac{n^{a}}{\ln(n)} + O\left(\frac{n^{a}}{\ln^{2}(n)}\right) \text{ [Finch, 2000]}$$

The *log*-average values

$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\frac{\ln P(i)}{\ln i} = \lambda$ [see Finch, 1999] $\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\frac{\ln S(i)}{\ln i} = \lambda$ [Finch, 1999]

and their generalizations

$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a} = \lambda_a$ [Shepp, 1964] $\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a} = \lambda_a$ [Finch, 2000].

The Harmonic Series

$\lim_{n\to\infty}\sum_{i=2}^{n}\frac{1}{S^{a}(i)} = \infty$ [Luca, 1999], [Tabirca, 1998] $\lim_{n\to\infty}\sum_{i=2}^{n}\frac{1}{P^{a}(i)} = \infty$ [Tabirca, 1999]

2. THE ERDOS-SMARANDACHE NUMBERS

From a combinatorial study of random permutation Sheep and Lloyd [1964] found the following integral equation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a} = \int_{0}^{\infty}\frac{x^{a-1}}{a!}\cdot\exp\left(-x-\int_{-x}^{\infty}\frac{\exp(-y)}{y}dy\right)dx := \lambda_a . \tag{7}$$

Finch [2000] started from Equation (7) and translated it from the Smarandache function.

Theorem [Finch, 2000] *If a is a positive integer number then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a}. \tag{8}$$

Proof

Many terms of the difference $\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a}$ are equal, therefore there will be reduced. This difference is transformed as follows:

$$\left|\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a}\right|=\frac{1}{n}\cdot\left|\sum_{i=2}^{n}\left[\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\left(\frac{\ln P(i)}{\ln i}\right)^{a}\right]\right|=$$

$$=\frac{1}{n}\cdot\left|\sum_{i:S(i)>P(i)}\left[\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\left(\frac{\ln P(i)}{\ln i}\right)^{a}\right]\right|\le\frac{1}{n}\cdot\sum_{i:S(i)>P(i)}\frac{\left|\ln^{a}S(i)-\ln^{a}P(i)\right|}{\ln^{a}i}.$$

The general term of the last sum is superiorly bounded by

$$\frac{\left|\ln^{a}S(i)-\ln^{a}P(i)\right|}{\ln^{a}i}\le\ln^{a}n$$

because $\left|\ln^{a}S(i)-\ln^{a}P(i)\right|=\ln^{a}S(i)-\ln^{a}P(i)\le\ln^{a}n$ and $\ln^{a}i>1$ ($i>3$).

Therefore, the chain of inequalities is continued as follows:

$$\left|\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a}-\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a}\right|\le\frac{1}{n}\cdot\ln^{a}n\cdot\left|i=\overline{1,n}:S(i)>P(i)\right|=$$

$$=\frac{1}{n}\cdot\ln^{a}n\cdot n\cdot e^{-(\sqrt{2}+a_n)\sqrt{\ln n\cdot\ln\ln n}}=\frac{\ln^{a}n}{e^{(\sqrt{2}+a_n)\sqrt{\ln n\cdot\ln\ln n}}}.$$

In order to prove that last right member tends to 0, we start from $\lim_{x\to\infty}\frac{x^{2a}}{e^{x}}=0$. We substitute $x=\sqrt{\ln n\cdot\ln\ln n}\to\infty$ and the limit becomes $\lim_{n\to\infty}\frac{(\ln n\cdot\ln\ln n)^{a}}{e^{\sqrt{\ln n\cdot\ln\ln n}}}=0$. Now, the last right member is calculated as follows:

$$\lim_{n\to\infty}\frac{\ln^{a}n}{e^{(\sqrt{2}+a_n)\sqrt{\ln n\cdot\ln\ln n}}}=\lim_{n\to\infty}\frac{(\ln n\cdot\ln\ln n)^{a}}{e^{\sqrt{\ln n\cdot\ln\ln n}}}\cdot\frac{1}{(\ln\ln n)^{a}}\cdot\frac{1}{e^{(\sqrt{2}-1+a_n)\sqrt{\ln n\cdot\ln\ln n}}}=0.$$

Therefore, the equation $\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^{a}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^{a}$ holds. ♣

The essence of this proof and the proof from [Finch, 2000] is given by Equation (6). But the above proof is a bit general giving even more

$$\lim_{n\to\infty}(\ln\ln n)^a \cdot \left[\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^a - \frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^a\right] = 0.$$

Definition. The Erdos-Smarandache number of order $a \in N$ is defined by

$$\lambda_a = \lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln S(i)}{\ln i}\right)^a = \lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\ln P(i)}{\ln i}\right)^a.$$

Equation (7) gives a formula for this number $\lambda_a = \int_0^\infty \frac{x^{a-1}}{a!}\cdot \exp\left(-x - \int_{-x}^{\infty}\frac{\exp(-y)}{y}dy\right)dx$. For a=1, we obtain that the Erdos-Smarandache number $\lambda_1 = \lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\frac{\ln S(i)}{\ln i} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=2}^{n}\frac{\ln P(i)}{\ln i}$, is in fact the Golomb-Dickman constant. Using a simple Maple computation the values of the first 20 Erdos-Smarandache numbers have been calculated with 15 exact decimals. They are presented below.

a=1 $\Rightarrow \lambda_1$=0.624329988543551
a=2 $\Rightarrow \lambda_2$=0.426695764659643
a=3 $\Rightarrow \lambda_3$=0.313630673224523
a=4 $\Rightarrow \lambda_4$=0.243876608021201
a=5 $\Rightarrow \lambda_5$=0.197922893443075
a=6 $\Rightarrow \lambda_6$=0.16591855680276
a=7 $\Rightarrow \lambda_7$=0.142575542115497
a=8 $\Rightarrow \lambda_8$=0.124890340441877
a=9 $\Rightarrow \lambda_9$=0.111067241922065
a=10 $\Rightarrow \lambda_{10}$=0.0999820620134543

a=11 $\Rightarrow \lambda_{11}$=0.0909016222187764
a=12 $\Rightarrow \lambda_{12}$=0.083330176072027
a=13 $\Rightarrow \lambda_{13}$=0.0769217248993612
a=14 $\Rightarrow \lambda_{14}$=0.0714279859927442
a=15 $\Rightarrow \lambda_{15}$=0.0666664107138031
a=16 $\Rightarrow \lambda_{16}$=0.0624998871487541
a=17 $\Rightarrow \lambda_{17}$=0.0588234792828849
a=18 $\Rightarrow \lambda_{18}$=0.0555555331402286
a=19 $\Rightarrow \lambda_{19}$=0.0526315688647356
a=20 $\Rightarrow \lambda_{20}$=0.049999954405103

3. Final Remarks

The numbers provided by Equation (7) could have many other names such as the Golomb-Dickman generalized constants or …. Because they are implied in Equation (8), we believe that a proper name for them is the Erdos-Smarandache numbers. We should also say that it is the Finch major contribution in rediscovering a quite old equation and connecting it with the Smarandache function.

References

Erdos, P. (1991) *Problem 6674*, Amer. Math. Monthly. **98**, 965.

Ford, K. (1999) The normal behaviours of the Smarandache function, *Smarandache Notions Journal,* Vol. **10,** No.1-2-3, 81-86.

Finch, S.R. (1999) The average of the Smarandache function, Smarandache Notions Journal, 10, 95-96. available at http://www.mathsoft.com/asolve/smarand/smarand.html.

Finch, S.R. (2000) Moments of the Smarandache function, [to be published in Smarandache Notions Journal], available at http://www.mathsoft.com/asolve/smarand/smarand.html.

Knuth, D.E. and Pardo, L.T. (1976) Analysis of a simple factorisation algorithm, *Theoretical Computer Science,* **3,** 321-348.

Luca, F. (1999) On the divergence of the Smarandache harmonic series, *Smarandache Notions Journal*, **10,** No. 1-2-3, 116-118.

Luca, F. (1999) The average value of the Smarandache function, [*Personal communication to Sabin Tabirca*].

Shepp, L.A. and Lloyd, S.P. (1966) Ordered cycle lengths in a random permutaion, *Trans. Amer. Math. Soc.,* **121,** 350-357.

Tabirca, S. and Tabirca, T. (1998) The convergence of the Smarandache harmonic series, *Smarandache Notions Journal*, **9**, No. 1-2, 27-35.

Tabirca, S. and Tabirca, T. (1999) The convergence of the Erdos harmonic series, *Smarandache Notions Journal*, **9**, No. 3-4-5.

Tabirca, S. and Tabirca, T. (1999a) The average value of the Erdos function, *Smarandache Notions Journal*, **9**, No. 3-4-5.

Smarandache, F. (1980) A Function in number theory, *Analele Univ. Timisoara*, **XVIII**.