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Vertex Graceful Labeling-Some Path Related Graphs

P.Selvaraju[1], P.Balaganesan[2] and J.Renuka[3]

[1]Department of Mathematics, Vel Tech Engineering College, Avadi, Chennai- 600 062, Tamil Nadu, India

[2]Department of Mathematics, School of Engineering, Saveetha University, Chennai- 602 105, Tamil Nadu, India

[3] Departments of Mathematics, Sai Ram College of Engineering, Chennai - 600 044, India

E-mail: pselvarr@gmail.com, balki2507@yahoo.co.in

Abstract: In this article, we show that an algorithm for VG of a caterpillar and proved that $A(m_j, n)$ is vertex graceful if m_j is monotonically increasing, $2 \leq j \leq n$, when n is odd, $1 \leq m_2 \leq 3$ and $m_1 < m_2$, $(m_j, n) \cup P_3$ is vertex graceful if m_j is monotonically increasing, $2 \leq j \leq n$, when n is odd, $1 \leq m_2 \leq 3$, $m_1 < m_2$ and $C_n \cup C_{n+1}$ is vertex graceful if and only if $n \geq 4$.

Key Words: Vertex graceful graphs, vertex graceful labeling, caterpillar, actinia graphs, Smarandachely vertex m-labeling.

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§1. Introduction

A graph G with p vertices and q edges is said to be vertex graceful if a labeling $f : V(G) \rightarrow \{1, 2, 3 \cdots p\}$ exists in such a way that the induced labeling $f^+ : E(G) \rightarrow Z_q$ defined by $f^+((u, v)) = f(u) + f(v)(mod\ q)$ is a bisection. The concept of vertex graceful (VG) was introduced by Lee, Pan and Tsai in 2005. Generally, if replacing q by an integer m and $f^S : E(G) \rightarrow Z_m$ also is a bijection, such a labeling is called a *Smarandachely vertex m-labeling*. Thus a vertex graceful labeling is in fact a Smarandachely vertex q-labeling.

All graphs in this paper are finite simple graphs with no loops or multiple edges. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G. The cardinality of the vertex set is called the order of G. The cardinality of the edge set is called the size of G. A graph with p vertices and q edges is called a (p, q) graph.

§2. Main Results

Algorithm 2.1

1. Let $v_1, v_2 \cdots v_n$ be the vertices of a path in the caterpillar. (refer Figure 1).
2. Let v_{ij} be the vertices, which are adjacent to v_i for $1 \leq i \leq n$ and for any j.
3. Draw the caterpillar as a bipartite graph in two partite sets denoted as Left (L) which

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contains $v_1, v_{2j}, v_3, v_{4j}, \cdots$ and for any j and Right (R) which contains $v_{1j}, v_2, v_{3j}, v_4, \cdots$ and for any j. (refer Figure 2).

4. Let the number of vertices in L be x.
5. Number the vertices in L starting from top down to bottom consecutively as $1, 2, \cdots, x$.
6. Number the vertices in R starting from top down to bottom consecutively as $(x+1), \cdots, q$. Note that these numbers are the vertex labels.
7. Compute the edge labels by adding them modulo q.
8. The resulting labeling is vertex graceful labeling.

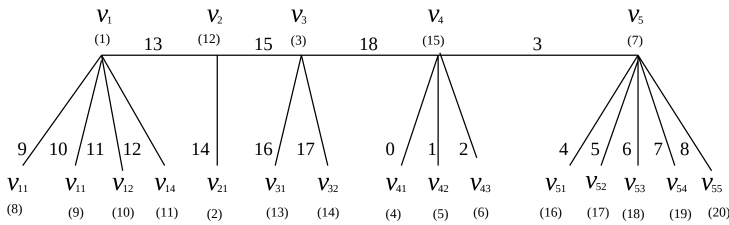

Figure 1: A caterpillar

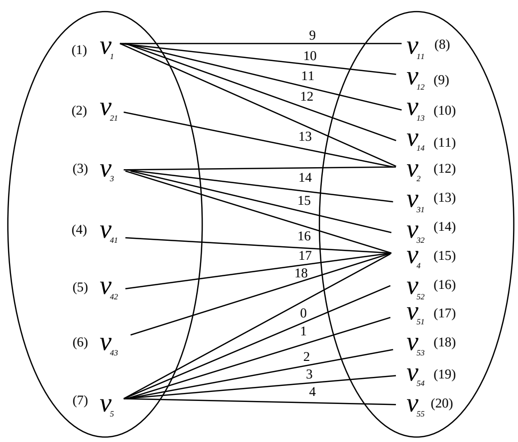

Figure 2: A caterpillar as bipartite graph

Definition 2.2 *The graph $A(m, n)$ obtained by attaching m pendent edges to the vertices of the cycle C_n is called Actinia graph.*

Theorem 2.3 *A graph $A(m_j, n), m_j$ is monotonically increasing with difference one, $2 \le j \le n$ is vertex graceful, $1 \le m_2 \le 3$ when n is odd.*

Proof Let the graph $G = A(m_j, n), m_j$ be monotonically increasing with difference one, $2 \le j \le n, n$ be odd with $p = n + m_n(\frac{m_n+1}{2}) - m_1(\frac{m_1+1}{2}), m_1 = m_2 - 1$ vertices and $q = p$ edges. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the cycle C_n. Let $v_{ij} (j = 1, 2, 3, \cdots, n)$ denote the vertices which are adjacent to v_i. By definition of vertex graceful labeling, the required vertices labeling are

$$v_i = \begin{cases} \dfrac{(i-1)}{2}\left(m_2 + \frac{(i+1)}{2}\right) + 1, 1 \le i \le n,\ i \text{ is odd},\\ (m_2+1)\frac{(n+1)}{2} + \left(\frac{n-1}{2}\right)^2 + \frac{(i-2)}{2}\left(m_2 + \frac{i}{2}\right) + \frac{i}{2}, 1 \le i \le n,\ i \text{ iseven}. \end{cases}$$

$$v_{ij} = \begin{cases} \dfrac{(n-1)}{2}\left(m_2 + \frac{(n+1)}{2}\right) + \frac{i-1}{2}\left(m_2 + \frac{i-3}{2}\right) + \frac{i+1}{2} + j, 1 \le j \le m_2 + i - 1, i \text{ is odd};\\ \dfrac{(i-2)}{2}\left(m_2 + \frac{i-2}{2}\right) + \frac{i}{2} + j, 1 \le i \le m_2 + i - 1, i \text{ is even}. \end{cases}$$

The corresponding edge set labels are as follows:

Let $A = \{e_i = v_i v_{i+1}/1 \le i \le n-1 \cup e_n = v_n v_1\}$, where

$$e_i = \left[\frac{(m_2+1)(n+1)}{2} + \left(\frac{n-1}{2}\right)^2 + m_2(i-1) + \frac{i(i+1)}{2} + 1\right] (\text{mod } q)$$

for $1 \le i \le n$. $B = \{e_{ij} = v_i v_{ij}/1 \le i \le n\}$, where

$$e_{ij} = \left[\frac{(n-1)}{2}\left(m_2 + \frac{(n+1)}{2}\right) + (i-1)\left(m_2 + \frac{i-1}{2}\right) + \frac{(i+1)}{2} + j + 1\right] (\text{mod } q)$$

for $1 \le i \le n$ and i is odd, $j = 1, 2, \cdots, m_2 + i - 1$. $C = \{e_{ij} = v_i v_{ij}/1 \le i \le n\}$, where

$$e_{ij} = \left[(m_2+1)\frac{(n+1)}{2} + \left(\frac{n-1}{2}\right)^2 + \frac{i-2}{2}(2m_2 + i - 1) + i + j\right] (\text{mod } q)$$

for $1 \le i \le n$ and i is even, $j = 1, 2, \cdots, m_2 + i - 1$.

Hence, the induced edge labels of G are q distinct integers. Therefore, the graph $G = A(m_j, n)$ is vertex graceful for n is odd, and $m \ge 1$. □

Theorem 2.4 *A graph $A(m_j, n) \cup P_3, m_j$ be monotonically increasing, $2 \le j \le n$ is vertex graceful, $1 \le m_2 \le 3, n$ is odd.*

Proof Let the graph $G = A(m_j, n) \cup P_3, m_j$ be monotonically increasing ,$2 \le j \le n$, n is odd with $p = n + 3 + m_n\frac{(m_n+1)}{2} - m_1\frac{(m_1+1)}{2}, m_1 < m_2$ vertices and $q = p - 1$ edges. Let$v_1, v_2, v_3, \cdots, v_n$ be the vertices of the cycle C_n. Let $v_{ij}(j = 1, 2, 3, \cdots, n)$ denote the vertices which are adjacent to v_i. Let u_1, u_2, u_3 be the vertices of the path P_3. By definition of vertex graceful labeling, the required vertices labeling are

$$v_i = \begin{cases} \frac{i-1}{2}\left(m_2 + \frac{i+1}{2}\right) + 1; 1 \le i \le n, i \text{ is odd};\\ (m_2+1)\frac{(n+1)}{2} + \left(\frac{n-1}{2}\right)^2 + \frac{(i-2)}{2}\left(m_2 + \frac{i}{2}\right) + \frac{i}{2} + 2; 1 \le i \le n, i \text{ is even}. \end{cases}$$

$$v_{ij} = \begin{cases} \frac{n-1}{2}\left(m_2 + \frac{n+1}{2}\right) + \frac{i-1}{2}\left(m_2 + \frac{i-3}{2}\right) + \frac{i+1}{2} + j + 2; 1 \le i \le n, i \text{ is odd},\\ \dfrac{i-2}{2}\left(m_2 + \frac{i-2}{2}\right) + \frac{i}{2} + j + 2; 1 \le i \le n, i \text{ is even}. \end{cases}$$

$u_i = \frac{n-1}{2}\left(m_2 + \frac{n+1}{2}\right) + \frac{i+1}{2} for\ \ i = 1, 3$ and $u_2 = p$.

The corresponding edge labels are as follows:

Let $A = \{e_i = v_i v_{i+1}/1 \le i \le n-1 \cup e_n = v_n v_1\}$, where

$$e_i = \left[\frac{(m_2+1)(n+1)}{2} + \left(\frac{n-1}{2}\right)^2 + m_2(i-1) + \frac{i(i+1)}{2} + 3\right] (\text{mod } q)$$

for $1 \leq i \leq n$. $B = \{e_{ij} = v_i v_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[\frac{(n-1)}{2}\left(m_2 + \frac{(n+1)}{2}\right) + (i-1)\left(m_2 + \frac{i-1}{2}\right) + \frac{(i+1)}{2} + j + 3\right] \pmod q$$

for $1 \leq i \leq n$ and i is odd, $j = 1, 2, \cdots, m_2 + i - 1$. $C = \{e_{ij} = v_i v_{ij}/1 \leq i \leq n\}$, where

$$e_{ij} = \left[(m_2+1)\frac{(n+1)}{2} + \left(\frac{n-1}{2}\right)^2 + \frac{i-2}{2}(2m_2 + i - 1) + i + j + 2\right] \pmod q$$

for1 $\leq i \leq n$ and i is even, $j = 1, 2, \cdots, m_2 + i - 1$. $D = \{e_i = u_i u_{i+1}$ for $i = 1, 2\}$, where

$$e_i = \left[\frac{n-1}{2}(m_2 + \frac{n+1}{2} + i + 1\right] \pmod q$$

for $i = 1, 2$. Hence, the induced edge labels of G are q distinct integers. Therefore, the graph $G = A(m_j, n) \cup P_3$ is vertex graceful for n is odd. □

Definition 2.5 *A regular lobster is defined by each vertex in a path is adjacent to the path P_2.*

Theorem 2.6 *A regular lobster is vertex graceful.*

Proof Let G be a 1- regular lobster with $3n$ vertices and $q = 3n - 1$ edges. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of a path P_n . Let v_i be the vertices, which are adjacent to v_{i1}^i and v_{i1}^i adjacent to v_{i2}^i for $1 \leq i \leq n$ and n is even .The theorem is proved by two cases. By definition of Vertex graceful labeling, the required vertices labeling are

Case 1 n is even

$$v_i = \begin{cases} \frac{3i-1}{2}; 1 \leq i \leq n, i \text{ is odd}, \\ \dfrac{3(n+i)}{2}; 1 \leq i \leq n, i \text{ is even}. \end{cases}$$

$$v_{i1} = \begin{cases} \dfrac{3(n+i)-1}{2} / \ 1 \leq i \leq n, i \text{ is odd} \\ \dfrac{3i-2)}{2} + 3 / \ 1 \leq i \leq n, i \text{ is even}. \end{cases}$$

$$v_{i2} = \begin{cases} \frac{3(i-1)}{2} + 2; 1 \leq i \leq n, i \text{ is odd}, \\ \dfrac{3(n+i)}{2} - 1; 1 \leq i \leq n, i \text{ is even}. \end{cases}$$

The corresponding edge labels are as follows:

Let $A = \{e_i = v_i v_{i+1}/1 \leq i \leq n-1\}$, where $e_i = \left(\dfrac{3(n+2i)}{2} + 1\right) \pmod q$ for $1 \leq i \leq n-1$, $B = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\dfrac{3(n+2i)}{2} - 1\right) \pmod q$ for $1 \leq i \leq n$ and i is odd, $C = \{e_{i1} = v_i v_{i1}/1 \leq i \leq n\}$, where $e_{i1} = \left(\dfrac{3(n+2i)}{2}\right) \pmod q$ for $1 \leq i \leq n$ and is even, $D = \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\dfrac{3(n+2i)}{2}\right) \pmod q$ for $1 \leq i \leq n$ and i is odd, $E = \{e_{i2} = v_{i1} v_{i2}/1 \leq i \leq n\}$, where $e_{i2} = \left(\dfrac{3(n+2i)}{2} - 1\right) \pmod q$ for $1 \leq i \leq n$ and is even.

Case 2 n is odd

$$v_i = \begin{cases} \frac{3i-1}{2};\ 1\le i\le n, i \text{ is odd},\\ \frac{3(n+i)+1}{2}; 1\le i\le n, i \text{ is even},\end{cases}$$

$$v_{i1} = \begin{cases} \frac{3(n+i)}{2}; 1\le i\le n, i \text{ is odd},\\ \frac{3(i-2)}{2}+3; 1\le i\le n, i \text{ is even},\end{cases}$$

$$v_{i2} = \begin{cases} \frac{3(i-1)}{2}+2; 1\le i\le n, i \text{ is odd},\\ \frac{3(n+i-1)}{2}+1; 1\le i\le n, i \text{ is even}.\end{cases}$$

The corresponding edge labels are determined by $A = \{e_i = v_iv_{i+1}/1 \le i \le n-1\}$, where $e_i = \left(\frac{3(n+2i+1)}{2}\right)$ (mod q) for $1 \le i \le n-1$, $B = \{e_{i1} = v_iv_{i1}/1 \le i \le n\}$, where $e_{i1} = \left(\frac{3(n+2i)-1}{2}\right)$ (mod q) for $1 \le i \le n$ and i is odd, $C = \{e_{i1} = v_iv_{i1}/1 \le i \le n\}$, where $e_{i1} = \left(\frac{3(n+2i)+1}{2}\right)$ (mod q) for $1 \le i \le n$ and is even, $D = \{e_{i2} = v_{i1}v_{i2}/1 \le i \le n\}$, where $e_{i2} = \left(\frac{3(n+2i)+1}{2}\right)$ (mod q) for $1 \le i \le n$ and i is odd, $E = \{e_{i2} = v_i1v_{i2}/1 \le i \le n\}$, where $e_{i2} = \left(\frac{3(n+2i)-1}{2}\right)$ (mod q) for $1 \le i \le n$ and is even. Hence the induced edge labels of G are q distinct edges. Therefore, the graph G is vertex graceful. □

Theorem 2.7 *$C_n \cup C_{n+1}$ is vertex graceful if and only if $n \ge 4$.*

Proof Let $G = C_n \cup C_{n+1}$ with $p = 2n+1$ vertices and $q = 2n+1$ edges. Suppose that the vertices of the cycle C_n run consecutively $u_1, u_2, \cdots, u_n$ with u_n joined to u_1 and that the vertices of the cycle C_{n+1} run consecutively $v_1, v_2, \cdots, v_{n+1}$ with v_{n+1} joined to v_1.

By definition of vertex graceful labeling

(a) $u_1 = 1, u_n = 2, u_i = 2i$ for $i = 2, 3, \cdots, \lfloor (n+1)/2 \rfloor, u_j = 2(n-j)+3$ for $j = \lfloor (n+3)/2 \rfloor, \cdots, n-1$.

(b) $v_1 = 2, v_2 = 2n-1$ and

(i) $v_{3s+t} = 2n-4t-6s+7, t = 0, 1, 2, s = 1, 2, \cdots, \lfloor (n+1-3t)/6 \rfloor$ if $s = \lfloor \frac{n+1-3t}{6} \rfloor < 1$ then no s.

(ii) Write $\alpha(0) = 0, \alpha(1) = 4, \alpha(2) = 2, \beta(0) = 0, \beta(1) = 3 = \beta(2)$
$v_{n+1-3s-t} = 2n-6s-\alpha(t), t = 0, 1, 2, s = 0, 1, \cdots, \lfloor \frac{n-5-\beta(t)}{6} \rfloor$. If $s = \lfloor \frac{n-5-\beta(t)}{6} \rfloor < 0$ then no s value exists.

(iii) We consider as that v_i to $f(i)$; and suppose that $n-2 = \theta\ mod(3), 0 \le \theta \le 2$. There are $2+\theta$ vertices as yet unlabeled. These middle vertices are labeled according to congruence class of modulo 6.

Congruence class	
n = 0 (mod 6)	f((n + 2)/2) = n + 2, f((n + 4)/2) = n + 3, f((n + 6)/2) = n + 4
n = 1 (mod 6)	f((n + 1)/2) = n + 2, f((n + 3)/2) = n + 3, f((n + 5)/2) = n + 4, f((n + 7)/2) = n+5
n = 2 (mod 6)	f((n + 2)/2) = n + 2, f((n + 4)/2) = n + 3
n = 2 (mod 6)	f((n + 1)/2) = n+4, f((n+3)/2) = n+3, f((n+5)/2) = n+2
n = 4 (mod 6)	f((n + 2)/2) = n+5, f((n+3)/2) = n+4, f((n+4)/2) = n+3, f((n+5)/2) = n+2
n = 4 (mod 6)	f((n + 3)/2) = n+3, f((n+5)/2) = n+2

To check that f is vertex graceful is very tedious. But we can give basic idea. The C_n cycle has edges with labels $\{2k+2/k=4,5,\cdots,n-1\}\cup\{0,3,5,7\}$. In this case all the labeling of the edges of the cycle C_{n+1} run consecutively v_1v_2 as follows:

$1,(2n-1,2n-3),(2n-11,2n-13,2n-15),\cdots,(2n+1-12k,2n-1-12k,2n-3-12k),\cdots,$ middle labels,$\cdots,(2n+3-12k,(2n+5-12k,(2n+7-12k),\cdots,(2n-21,2n-19,2n-17),(2n-9,2n-7,2n-5),2$. The middle labels depend on the congruence class modulo and are best summarized in the following table. If n is small the terms in brackets alone occur.

Congruence class	
$n=0(\text{mod}6)$	$\cdots(11,9),6,4,7,(13,15,17)\cdots$
$n=1(\text{mod}6)$	$\cdots(13,11),6,4,7,(13,15,17)\cdots$
$n=2(\text{mod}6)$	$\cdots(11),6,4,7,(9)\cdots$
$n=2(\text{mod}6)$	$\cdots(13),7,4,6,(9,11)\cdots$
$n=4(\text{mod}6)$	$\cdots(15,9),6,4,7(11,13)\cdots$
$n=4(\text{mod}6)$	$\cdots(9),7,6,4(11,13,15)\cdots$

Thus, all these edge labelings are distinct. □

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