Scientia Magna
Vol. 1 (2005), No. 2, 96-108

On the number of Smarandache zero-divisors and Smarandache weak zero-divisors in loop rings

W.B.Vasantha and Moon K.Chetry

Department of Mathematics
I.I.T.Madras,Chennai

Abstract In this paper we find the number of smarandache zero divisors (S-zero divisors) and smarandache weak zero divisors (S-weak zero divisors) for the loop rings $Z_2L_n(m)$ of the loops $L_n(m)$ over Z_2. We obtain the exact number of S-zero divisors and S-weak zero divisors when $n = p^2$ or p^3 or pq where p, q are odd primes. We also prove $ZL_n(m)$ has infinitely many S-zero divisors and S-weak zero divisors, where Z is the ring of integers. For any loop L we give conditions on L so that the loop ring Z_2L has S-zero divisors and S-weak zero divisors.

§0 . Introduction

This paper has four sections. In the first section, we just recall the definitions of S-zero divisors and S-weak zero divisors and some of the properties of the new class of loops $L_n(m)$. In section two, we obtain the number of S-zero divisors of the loop rings $Z_2L_n(m)$ and show when $n = p^2$, where p is an odd prime, $Z_2L_n(m)$ has $p(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Also when $n = p^3$, p an odd prime, $Z_2L_n(m)$ has $p(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r) + p^2(1 + \sum_{r=2,reven}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Again when $n = pq$, where p, q are odd primes, $Z_2L_n(m)$ has $p+q+p(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r)+q(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-zero divisors. Further we prove $ZL_n(m)$ has infinitely many S-zero divisors. In section three, we find the number of S-weak zero divisors for the loop ring $Z_2L_n(m)$ and prove that when $n = p^2$, where p is an odd prime, $Z_2L_n(m)$ has $2p(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-weak zero divisors. Also when $n = p^3$, where p is an odd prime, $Z_2L_n(m)$ has $2p(\sum_{r=2,reven}^{p^2-1} {}^{p^2+1}C_r) + 2p^2(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$ S-weak zero divisors. Again when $n = pq$, where p, q are odd primes, $Z_2L_n(m)$ has $2[p(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r) + q(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)]$ S-weak zero divisors. We prove $ZL_n(m)$ has infinitely many S-weak zero divisors. The final section gives some unsolved problems and some conclusions based on our study.

§1. Basic Results

Here we just recollect some basic results to make this paper a self contained one.

Definition 1.1[4]. Let R be a ring. An element $a \in R\backslash\{0\}$ is said to be a S-zero divisor if $a.b = 0$ for some $b \neq 0$ in R and there exists $x, y \in R\backslash\{0, a, b\}$ such that

$$i. \quad a.x = 0 \quad or \quad x.a = 0$$

$$ii \quad b.y = 0 \quad or \quad y.b = 0$$

$$iii. \quad x.y \neq 0 \quad or \quad y.x \neq 0$$

Definition 1.2[4]. Let R be a ring. An element $a \in R\backslash\{0\}$ is a S-weak zero divisor if there exists $b \in R\backslash\{0, a\}$ such that $a, b = 0$ satisfying the following conditions: There exists $x, y \in R\backslash\{0, a, b\}$ such that

$$i. \quad a.x = 0 \quad or \quad x.a = 0$$

$$ii. \quad b.y = 0 \quad or \quad y.b = 0$$

$$iii. \quad x.y = 0 \quad or \quad y.x = 0$$

Definition 1.3[3]. Let $L_n(m) = \{e, 1, 2, 3 \cdots, n\}$ be a set where $n > 3$, n is odd and m is a positive integer such that $(m, n) = 1$ and $(m - 1, n) = 1$ with $m < n$. Define on $L_n(m)$, a binary operation $'.'$ as follows:

$$i. \quad e.i = i.e \quad for \quad all \quad i \in L_n(m)\backslash\{e\}$$

$$ii. \quad i^2. = e \quad for \quad all \quad i \in L_n(m)$$

$$iii. \quad i.j = t, \quad where \quad t \equiv (mj-(m-1)i)(mod\, n) \quad for \quad all \quad i, j \in L_n(m), \quad i \neq e \quad and \quad j \neq e.$$

Then $L_n(m)$ is a loop. This loop is always of even order; further for varying m, we get a class of loops of order $n + 1$ which we denote by L_n.

Example 1.1[3]. Consider $L_5(2) = \{e, 1, 2, 3, 4, 5\}$. The composition table for $L_5(2)$ is given below:

.	e	1	2	3	4	5
e	e	1	2	3	4	5
1	1	e	3	5	2	4
2	2	5	e	4	1	3
3	3	4	1	e	5	2
4	4	3	5	2	e	1
5	5	2	4	1	3	e

This loop is non-commutative and non-associative and of order 6.

Theorem 1.1[3]. Let $L_n(m) \in L_n$. For every $t|n$ there exists t subloops of order $k + 1$, where $k = n/t$.

Theorem 1.2[3]. Let $L_n(m) \in L_n$. If H is a subloop of $L_n(m)$ of order $t+1$, then $t|n$.

Remark 1.2[3]. Lagrange's theorem is not satisfied by all subloops of the loop $L_n(m)$,i.e there always exists a subloop H of $L_n(m)$ which does not satisfy the Lagrange's theorem, i.e $o(H) \dagger o(L_n(m))$.

§2. Definition of the number of S-zero divisors in $Z_2L_n(m)$ and $ZL_n(m)$

In this section, we give the number of S-zero divisors in $Z_2L_n(m)$. We prove $ZL_n(m)$ (where $n = p^2$ or pq, p and q are odd primes), has infinitely many S-zero divisors. Further we show any loop L of odd (or even) order if it has a proper subloop of even (or odd) order then the loop ring $Z_2L_n(m)$ over the field Z_2 has S-zero divisors. We first show if L is a loop of odd order and L has a proper subloop of even order, then $Z_2L_n(m)$ has S-zero divisors.

Theorem 2.1. Let L be a finite loop of odd order. $Z_2 = \{0, 1\}$, the prime field of characteristic 2. Suppose H is a subloop of L of even order, then Z_2L has S-zero divisors.

Proof. Let $|L| = n$; where n is odd. Z_2L be the loop ring of L over Z_2. H be the subloop of L of order m, where m is even. Let $X = \sum_{i=1}^{n} g_i$ and $Y = \sum_{i=1}^{m} h_i$, then

$$X.Y = 0.$$

Now

$$(1 + g_t)X = 0, \quad g_t \in l\backslash H.$$

also

$$(1 + h_i + h_j + h_k)Y = 0, \quad h_i, h_j, h_k \in H.$$

so that

$$(1 + g_t)(1 + h_i + h_j + h_k) \neq 0.$$

Hence the claim.

Corollary 2.1. If L is a finite loop of even order n and H is a subloop of odd order m, then the loop ring Z_2L has S-zero divisors.

It is important here to mention that Z_2L may have other types of S-zero divisors. This theorem only gives one of the basic conditions for Z_2L to have S-zero divisors.

Example 2.1. Let $Z_2L_{25}(m)$ be the loop ring of the loop $L_{25}(m)$ over Z_2, where $(m, 25) = 1$ and $(m - 1, 25) = 1$. As $5|25$, so $L_{25}(m)$ has 5 proper subloops each of order 6. Let H be one of the proper subloops of $L_{25}(m)$.

Now take

$$X = \sum_{i=1}^{26} g_i, \quad Y = \sum_{i=1}^{6} h_i, \quad g_i \in L_{25}(m), \quad h_i \in H,$$

then

$$(1 + g_i)X = 0, \quad g_i \in L_{25}(m)\backslash H$$

$$(1+h_i)Y=0, \quad h_i \in H$$

but

$$(1+g_i)(1+h_i) \neq 0.$$

so X and Y are S-zero divisors in $Z_2L_{25}(m)$.

Theorem 2.2. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number,$n>3$) with $n=p^2$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p\left(1+\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-zero divisors.

Proof. Given $L_n(m)$ is a loop of order $n+1$, where $n=p^2$ (p an odd prime). Let $Z_2L_n(m)$ be the loop ring of the loop $L_n(m)$ over Z_2. Now clearly the loop $L_n(m)$ has exactly p subloops of order $p+1$. The number of S-zero divisors in $Z_2L_n(m)$ for $n=p^2$ can be enumerated in the following way: Let

$$X=\sum_{i=1}^{n+1} g_i \quad and \quad Y=\sum_{i=1}^{p+1} h_i$$

where $g_i \in L_n(m)$ and $h_i \in H_j$. For this

$$X.Y=0$$

choose

$$a=(1+g), \quad g \in L_n(m) \backslash H_j$$

$$b=(h_i+h_j), \quad h_i, h_j \in H_j$$

then

$$a.X=0 \quad and \quad b.Y=0$$

but

$$a.b \neq 0.$$

So X and Y are S-zero divisors. There are p such S-zero divisors, as we have p subloops H_j ($j=1,2,\cdots,p$) of $L_n(m)$.

Next consider, S-zero divisors of the form

$$(h_1+h_2)\sum_{i=1}^{n+1} g_i=0, \quad where \quad h_1, h_2 \in H_j, \quad g_i \in L_n(m)$$

put

$$X=(h_1+h_2), \quad Y=\sum_{i=1}^{n+1} g_i$$

we have ${}^{p+1}C_2$ such S-zero divisors. This is true for each of the subloops. Hence there exists ${}^{p+1}C_2 \times p$ such S-zero divisors. Taking four elements h_1, h_2, h_3, h_4 from H_j at a time, we get

$$(h_1+h_2+h_3+h_4)\sum_{i=1}^{n+1} g_i=0$$

so we get ${}^{p+1}C_4 \times p$ such S-zero divisors. Continue in this way, we get

$$(h_1 + h_2 + \cdots + h_{p-1}) \sum_{i=1}^{n+1} g_i = 0, \quad where \quad h_1, h_2, \cdots, h_{p-1} \in H_j$$

So we get ${}^{p+1}C_{p-1} \times p$ such S-zero divisors. Adding all these S-zero divisors, we get

$$p \left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r \right)$$

number of S-zero divisors in the loop ring $Z_2 L_n(m)$. Hence the claim.

Example 2.2. Let $Z_2 L_{49}(m)$ be the loop ring of the loop $L_{49}(m)$ over Z_2, where $(m, 49) = 1$ and $(m - 1, 49) = 1$. Here $p = 7$, so from Theorem 2.2, $Z_2 L_{49}(m)$ has

$$7 \left(1 + \sum_{r=2,\ r\ even}^{6} {}^{7+1}C_r \right)$$

S-zero divisors i.e $7(1 + \sum_{r=2,\ r\ even}^{6} {}^{8}C_r) = 889$ S-zero divisors.

Theorem 2.3. Let $L_n(m)$ be a loop of order $n + 1$ (n an odd number, $n > 3$) with $n = p^3$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2 L_n(m)$ has exactly

$$p \left(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r \right) + p^2 \left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r \right)$$

S-zero divisors.

Proof. We enumerate all the S-zero divisors of $Z_2 L_n(m)$ in the following way:

Case I: As $p | p^3, L_n(m)$ has p proper subloops H_j each of order $p^2 + 1$. In this case I, we have $p^2 - 1$ types of S-zero divisors. We just index them by type I_1, type I_2, $\cdots$, type I_{p^2-1}.

Type I_1: Here

$$\sum_{i=1}^{n+1} g_i \sum_{i=1}^{p^2+1} h_i = 0, \quad g_i \in L_n(m), \quad h_i \in H_j, (j = 1, 2, \cdots, p)$$

So we will get p S-zero divisors of this type.

Type I_2:

$$(h_1 + h_2) \sum_{i=1}^{n+1} g_i = 0, \quad h_1, h_2 \in H_j (j = 1, 2, \cdots, p).$$

As in the Theorem 2.2, we will get ${}^{p^2+1}C_2 \times p$ S-zero divisors of this type.

Type I_3:

$$(h_1 + h_2 + h_3 + h_4) \sum_{i=1}^{n+1} g_i = 0, \quad h_1, h_2, h_3, h_4 \in H_j (j = 1, 2, \cdots, p).$$

We will get ${}^{p^2+1}C_4 \times p$ S-zero divisors of this type.

Continue this way,

Type I_{p^2-1}:

$$(h_1 + h_2 + \cdots + h_{p^2-1}) \sum_{i=1}^{n+1} g_i = 0, \quad h_i \in H_j$$

We will get ${}^{p^2+1}C_{p^2-1} \times p$ S-zero divisors of this type. Hence adding all this types of S-zero divisors we will get

$$p\left(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right)$$

S-zero divisors for case I.

Case II: Again $p^2|p^3$, so there are p^2 subloops H_j each of order $p+1$. Now we can enumerate all the S-zero divisors in this case exactly as in case I above. So there are

$$p^2(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r)$$

S-zero divisors. Hence the total number of S-zero divisors in $Z_2L_n(m)$ is

$$p\left(1 + \sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right) + p^2\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

Hence the claim.

Example 2.3. Let $Z_2L_{27}(m)$ be the loop ring of the loop $L_{27}(m)$ over Z_2, where $(m, 27) = 1$ and $(m-1, 27) = 1$. Here $p = 3$, so from Theorem 2.3, $Z_2L_{27}(m)$ has

$$3(1 + \sum_{r=2,\ r\ even}^{8} {}^{3^2+1}C_r) + 3^2(1 + \sum_{r=2,\ r\ even}^{2} {}^{4}C_r)$$

S-zero divisors i.e $3\left(1 + \sum_{r=2,\ r\ even}^{8} {}^{10}C_r\right) + 9\left(1 + \sum_{r=2,\ r\ even}^{2} {}^{4}C_r\right) = 1533$ S-zero divisors.

Theorem 2.4. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = pq$, where p, q are odd primes. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$p + q + p\left(1 + \sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right) + q\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-zero divisors.

Proof. We will enumerate all the S-zero divisors in the following way:

Case I: As $p|pq$, $L_n(m)$ has p subloops H_j each of order $q+1$. Proceeding exactly in the same way as in the Theorem 2.3, we will get $p + p\left(1 + \sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)$ S-zero divisors for case I.

Case II: Again $q|pq$, so $L_n(m)$ has q subloops H_j each of order $p+1$. Now as above we will get $q + q\left(1 + \sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$ S-zero divisors for case II. Hence adding all the S-zero divisors in case I and case II, we get

$$p+q+p\left(1+\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)+q\left(1+\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-zero divisors in $Z_2L_n(m)$.

Hence the claim.

Now we prove for the loop ring $ZL_n(m)$ when $n = p^2$ or p^3 or pq, where p, q are odd primes, $ZL_n(m)$ has infinitely many S-zero divisors.

Theorem 2.5. Let $ZL_n(m)$ be the loop ring of the loop $L_n(m)$ over Z, where $n = p^2$ or p^3 or pq (p, q are odd primes), then $ZL_n(m)$ has infinitely many S-zero divisors.

Proof. Let $L_n(m)$ be a loop ring such that $n = p^2$. $L_n(M)$ has p subloops (say H_j) each of order $p+1$.

Now the loop ring $ZL_n(m)$ has the following types of S-zero divisors:

$$X = a - bh_1 + bh_2 - ah_3 \quad and \quad Y = \sum_{i=1}^{n+1} g_i$$

where $a, b \in Z$ and $h_i \in H_i$, $g_i \in L_n(m)$ such that

$$(a - bh_1 + bh_2 - ah_3)\sum_{i=1}^{n+1} g_i = 0$$

Again

$$(1 - g_k)Y = 0, \quad g_k \in L_n(m)\backslash H_j$$

also

$$(a - bh_1 + bh_2 - ah_3)\sum h_i = 0, \quad h_i \in H_j$$

clearly

$$(1-g_k)\left(\sum_{h_i \in H_j} h_i\right) \neq 0.$$

So X, Y are S-zero divisors in $ZL_n(m)$. Now we see there are infinitely many S-zero divisors of this type for a and b can take infinite number of values in Z. For $n = p^2$ or p^3 or pq we can prove the results in a similar way. Hence the claim.

§3. Determination of the number of S-weak zero divisors in $Z_2L_n(m)$ and $ZL_n(m)$

In this section, we give the number of S-weak zero divisors in the loop ring $Z_2L_n(m)$ when n is of the form p^2, p^3 or pq where p and q are odd primes. Before that we prove the existence of S-weak zero divisors in the loop ring Z_2L whenever L has a proper subloop.

Theorem 3.1. Let n be a finite loop of odd order. Suppose H is a subloop of L of even order, then Z_2L has S-weak zero divisors.

Proof. Let $|L| = n$; n odd. Z_2L be the loop ring. H be the subloop of L of order m, where m is even. Let $X = \sum_{i=1}^{n} g_i$ and $Y = 1 + h_t, g_i \in L, h_t \in H$, then

$$X.Y = 0$$

Now

$$Y.\sum_{i=1}^{m} h_i = 0, \quad h_i \in H$$

also

$$X(1 + g_t) = 0, \quad g_t(\neq h_t) \in H$$

so that

$$(1 + g_t)\sum_{i=1}^{m} h_i = 0.$$

Hence the claim.

Example 3.1. Let $Z_2L_{25}(m)$ be the loop ring of the loop $L_{25}(m)$ over Z_2, where $(m, 25) = 1$ and $(m - 1, 25) = 1$. As $5|25$, so $L_{25}(m)$ has 5 proper subloops each of order 6.

Take

$$X = \sum_{i=1}^{26} g_i, \quad Y = 1 + h_t, \quad g_i \in L_{25}(m), \quad h_t \in H$$

then

$$X.Y = 0$$

again

$$X(1 + g_t) = 0, \quad g_t(\neq h_t) \in H$$

$$Y\sum_{i=1}^{6} h_i = 0, \quad h_i \in H$$

also

$$(1 + g_t)\sum_{i=1}^{6} h_i = 0,$$

So X and Y are S-weak zero divisors in $Z_2L_{25}(m)$.

Example 3.2. Let $Z_2L_{21}(m)$ be the loop ring of the loop $L_{21}(m)$ over Z_2, where where $(m, 21) = 1$ and $(m - 1, 21) = 1$. As $3|21$, so $L_{21}(m)$ has 3 proper subloops each of order 8.

Take

$$X = \sum_{i=1}^{8} h_i, \quad Y = 1 + h_t, \quad h_i, h_t \in H$$

then

$$X.Y = 0$$

again

$$X(1 + g_t) = 0, \quad g_t(\neq h_t) \in H$$

$$Y\sum_{i=1}^{22} g_i = 0, \quad g_i \in L_{21}(m)$$

also

$$(1 + g_t)\sum_{i=1}^{22} g_i = 0,$$

So X and Y are S-weak zero divisors in $Z_2L_{21}(m)$.

Theorem 3.2. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = p^2$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2p\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-weak zero divisors.

Proof. Clearly the loop $L_n(m)$ has p subloops H_j each of order $p+1$. As in case of Theorem 2.3, we index the $p-1$ types of S-weak zero divisors by $I_1, I_2, \cdots, I_{p-1}$. Now the number of S-weak zero divisors in $Z_2L_n(m)$ for $n = p^2$ can be enumerated in the following way:

Type I_1. Let

$$X = h_1 + h_2, \quad Y = \sum_{i=1}^{n+1} g_i$$

where $h_1, h_2 \in H_j$ and $g_i \in L_n(m)$ then

$$XY = 0$$

take

$$a = \sum_{i=1}^{p+1} h_i, \quad and \quad b = h_3 + h_4 \quad where \quad h_i \in H_j, \quad (j = 1, 2, \cdots, p)$$

then

$$aX = 0, \quad bY = 0$$

also

$$ab = 0$$

So for each proper subloop we will get ${}^{p+1}C_2$ S-weak zero divisors and as there are p proper subloops we will get ${}^{p+1}C_2 \times p$ such S-weak zero divisors.

Type I_2. Again let

$$X = h_1 + h_2, \quad Y = \sum_{i=1}^{p+1} h_i, \quad h_i \in H_j$$

then

$$XY = 0$$

take

$$a = \sum_{i=1}^{n+1} g_i, \quad g_i \in L_n(m), \quad b = h_1 + h_2, \quad h_1, h_2 \in H_j,$$

then

$$aX = 0, \quad bY = 0$$

also

$$ab = 0$$

Here also we will get ${}^{p+1}C_2 \times p$ such S-weak zero divisors of this type.

Type I_3.

$$(h_1 + h_2 + h_3 + h_4) \sum_{i=1}^{n+1} g_i, \quad g_i \in L_n(m), \quad h_i \in H_j.$$

As above we can say there are ${}^{p+1}C_4 \times p$ such S-weak zero divisors.

Type I_4.

$$(h_1 + h_2 + h_3 + h_4) \sum_{i=1}^{p+1} h_i, \quad h_i \in H_j.$$

There are ${}^{p+1}C_4 \times p$ such S-weak zero divisors.

Continue this way,

Type I_{p-2}.

$$(h_1 + h_2 + \cdots + h_{p-1}) \sum_{i=1}^{n+1} g_i, \quad g_i \in L_n(m), \quad h_i \in H_j.$$

there are ${}^{p+1}C_{p-1} \times p$ such S-weak zero divisors.

Type I_{p-1}.

$$(h_1 + h_2 + \cdots + h_{p-1}) \sum_{i=1}^{n} h_i, \quad h_i \in H_j.$$

Again there are ${}^{p+1}C_{p-1} \times p$ such S-weak zero divisors of this type. Adding all these S-weak zero divisors we will get the total number of S-weak zero divisors in $Z_2L_n(m)$ as

$$2p \left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r \right)$$

Hence the claim.

Theorem 3.3. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = p^3$, p an odd prime. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2p \left(\sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r \right) + 2p^2 \left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r \right)$$

S-weak zero divisors.

Proof. We enumerate all the S-weak zero divisors of $Z_2L_n(m)$ in the following way:

Case I: As $p|p^3$, $L_n(m)$ has p proper subloops H_j each of order $p^2 + 1$. Now as in the Theorem 3.2.

Type I_1:

$$(h_1 + h_2) \sum_{i=1}^{n+1} g_i = 0, \quad g_i \in L_n(m), \quad h_i \in H_j.$$

So we will get ${}^{p^2+1}C_2 \times p$ S-weak zero divisors of type I_1.

Type I_2:

$$(h_1 + h_2) \sum_{i=1}^{p^2+1} h_i = 0, \quad h_i \in H_j.$$

So we will get ${}^{p^2+1}C_2 \times p$ S-weak zero divisors of type I_2.

Continue in this way

Type I_{p^2-2}:

$$(h_1 + h_2 + \cdots + h_{p^2-1}) \sum_{i=1}^{n+1} g_i = 0,$$

So we will get ${}^{p^2+1}C_{p^2-1} \times p$ S-weak zero divisors of this type.

Type I_{p^2-1}:

$$(h_1 + h_2 + \cdots + h_{p^2-1}) \sum_{i=1}^{p^2+1} h_i = 0,$$

So we will get ${}^{p^2+1}C_{p^2-1} \times p$ S-weak zero divisors of type I_{p^2-1}.

Adding all this S-weak zero divisors, we will get the total number of S-weak zero divisors (in case I) in $Z_2L_n(m)$ as $2p\left(\sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right)$.

Case II: Again $p^2|p^3$, so there are p^2 proper subloops H_j each of order $p+1$. Now we can enumerate all the S-weak zero divisors in this case exactly as in case I above. So there are

$$2p^2\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-weak zero divisors in case II.

Hence the total number of S-weak zero divisors in $Z_2L_n(m)$ is

$$2p\left(\sum_{r=2,\ r\ even}^{p^2-1} {}^{p^2+1}C_r\right) + 2p^2\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

Hence the claim.

Theorem 3.4. Let $L_n(m)$ be a loop of order $n+1$ (n an odd number, $n > 3$) with $n = pq$, p, q are odd primes. Z_2 be the prime field of characteristic 2. The loop ring $Z_2L_n(m)$ has exactly

$$2\left[p\left(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right) + q\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)\right]$$

S-weak zero divisors.

Proof. We will enumerate all the S-weak zero divisors in the following way:

Case I: As $p|pq, L_n(m)$ has p proper subloops H_j each of order $q+1$. Proceeding exactly same way as in Theorem 3.3, we will get

$$2p\left(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)$$

S-weak zero divisors in case I.

Case II: Again as $q|pq, L_n(m)$ has q proper subloops H_j each of order $p+1$. So as above we will get

$$2q\left(\sum_{r=2,\ r\ even}^{p-1} {}^{p+1}C_r\right)$$

S-weak zero divisors in case II.

Hence adding all the S-weak zero divisors in case I and case II, we get

$$2\left[p\left(\sum_{r=2,\ r\ even}^{q-1} {}^{q+1}C_r\right)+q\left(\sum_{r=2,\ r4\ even}^{p-1} {}^{p+1}C_r\right)\right]$$

S-weak zero divisors in $Z_2L_n(m)$.

Hence the claim.

Now we prove for the loop ring $ZL_n(m)$ where $n=p^2$ or p^3 or pq, (p,q are odd primes), $ZL_n(m)$ has infinitely many S-weak zero divisors.

Theorem 3.5. Let $ZL_n(m)$ be the loop ring of the loop $L_n(m)$ over Z, where $n=p^2$ or p^3 or pq (p,q are odd primes), then $ZL_n(m)$ has infinitely many S-weak zero divisors.

Proof. Let $L_n(m)$ be a loop ring such that $n=p^2$. $L_n(M)$ has p subloops (say H_j) each of order $p+1$. Now the loop ring $ZL_n(m)$ has the following types of S-weak zero divisors:

$$X=a-bh_1+bh_2-ah_3 \quad and \quad Y=\sum_{i=1}^{n+1} g_i$$

where $a,b\in Z, g_i\in L_n(m)$ and $h_1,h_2,h_3\in H_j$ are such that

$$XY=0.$$

Again

$$X\sum_{i=1}^{p+1} h_i=0, \quad h_i\in H_j$$

also

$$(1-g_t)Y=0, \quad g_t(\neq h_t)\in H_j$$

clearly

$$(1-g_t)\left(\sum_{i=1}^{p+1} h_i\right)=0.$$

So X,Y are S-weak zero divisors in $ZL_n(m)$. Now we see there are infinitely many S-weak zero divisors of this type for a and b can take infinite number of values in Z.

For $n=p^2$ or p^3 or pq we can prove the results in a similar way.

Hence the claim.

§4. Conclusions:

In this paper we find the exact number of S-zero divisors and S-weak zero divisors for the loop rings $Z_2L_n(m)$ in case of the special type of loops $L_n(m) \in L_n$ over Z_2, when $n = p^2$ or p^3 or pq (p, q are odd primes). We also prove for the loop ring $ZL_n(m)$ has infinite number of S-zero divisors and S-weak zero divisors. We obtain conditions for any loop L to have S-zero divisors and S-weak zero divisors. We suggest it would be possible to enumerate in the similar way the number of S-zero divisors and S-weak zero divisors for the loop ring $Z_2L_n(m)$ when $n = p^s, s > 3$; p a prime or when $p = p_1p_2 \cdots p_t$ where $p_1, p_2, \cdots, p_t$ are odd primes. However we find it difficult when we take Z_p instead of Z_2, where p can be odd prime or a composite number such that $(p, n+1 = 1)$ or $(p, n+1 = p)$ and n is of the form $n = p_1^{t_1}p_2^{t_2} \cdots p_r^{t_r}, t_i > 1, n$ is odd and $p_1, p_2, \cdots p_r$ are odd primes.

References

[1] R.H. Bruck, A survey of binary system, Spinger Verlag (1958).

[2] D.S.Passman, The algebraic structure of group rings, Wiley interscience, (1977).

[3] S.V.Singh, On a new class of loops and loop rings, PhD thesis, IIT Madras, (1994).

[4] Vasantha Kandasamy,W.B, Smarandache Zero divisors, (2001).
http://www.gallup.unm.edu/smarandache/Zero-divisor.pdf