

Simple Path Covers in Graphs

S. Arumugam and I. Sahul Hamid

National Centre for Advanced Research in Discrete Mathematics of
Kalasalingam University, Anand Nagar, Krishnankoil-626190, INDIA.
E-mail: *s_arumugam_akce@yahoo.com*

Abstract: A simple path cover of a graph G is a collection ψ of paths in G such that every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common. More generally, for any integer $k \geq 1$, a Smarandache path k -cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ , then a Smarandache path k -cover of G is a simple path cover of G . The minimum cardinality of a simple path cover of G is called the simple path covering number of G and is denoted by $\pi_s(G)$. In this paper we initiate a study of this parameter.

Key Words: Smarandache path k -cover, simple path cover, simple path covering number.

AMS(2000): 05C35, 05C38.

§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [5]. All graphs in this paper are assumed to be connected and non-trivial.

If $P = (v_0, v_1, v_2, \dots, v_n)$ is a path or a cycle in a graph G , then v_1, v_2, \dots, v_{n-1} are called internal vertices of P and v_0, v_n are called external vertices of P . If $P = (v_0, v_1, v_2, \dots, v_n)$ and $Q = (v_n = w_0, w_1, w_2, \dots, w_m)$ are two paths in G , then the walk obtained by concatenating P and Q at v_n is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \dots, v_2, v_1, v_0)$ is denoted by P^{-1} . For a unicyclic graph G with cycle C , if w is a vertex of degree greater than 2 on C with $\deg w = k$, let e_1, e_2, \dots, e_{k-2} be the edges of $E(G) - E(C)$ incident with w . Let $T_i, 1 \leq i \leq k-2$, be the maximal subtree of G such that T_i contains the edge e_i and w is a pendant vertex of T_i . Then T_1, T_2, \dots, T_{k-2} are called the *branches* of G at w . Also the maximal subtree T of G such that $V(T) \cap V(C) = \{w\}$ is called the *subtree* rooted at w .

The concept of path cover and path covering number of a graph was introduced by Harary [6]. Preliminary results on this parameter were obtained by Harary and Schwenk [7], Peroche [9] and Stanton et al. [10], [11].

¹Supported by NBHM Project 48/2/2004/R&D-II/7372.

²Received May 16, 2008. Accepted September 16, 2008.

Definition 1.1([6]) *A path cover of a graph G is a collection ψ of paths in G such that every edge of G is in exactly one path in ψ . The minimum cardinality of a path cover of G is called the path covering number of G and is denoted by $\pi(G)$ or simply π .*

Theorem 1.2([10]) *For any tree T with k vertices of odd degree, $\pi(T) = \frac{k}{2}$.*

Theorem 1.3([7]) *The path covering number of the complete graph K_p is given by $\pi(K_p) = \lceil \frac{p}{2} \rceil$. (For any real number x , $\lceil x \rceil$ denotes the least positive integer $\geq x$.)*

Theorem 1.4([4]) *Let G be a unicyclic graph with unique cycle C . Let m denote the number of vertices of degree greater than 2 on C . Let k denote the number of vertices of odd degree. Then*

$$\pi(G) = \begin{cases} 2 & \text{if } m = 0 \\ \frac{k}{2} + 1 & \text{if } m = 1 \\ \frac{k}{2} & \text{otherwise} \end{cases}$$

Theorem 1.5([4]) *For any graph G , $\pi(G) \geq \lceil \frac{\Delta}{2} \rceil$.*

The concepts of graphoidal cover and acyclic graphoidal cover were introduced by Acharya et al. [1] and Arumugam et al. [4].

Definition 1.6([1]) *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions.*

- (i) *Every path in ψ has at least two vertices.*
- (ii) *Every vertex of G is an internal vertex of at most one path in ψ .*
- (iii) *Every edge of G is in exactly one path in ψ .*

If further no member of ψ is a cycle in G , then ψ is called an acyclic graphoidal cover of G . The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$. Similarly we define the acyclic graphoidal covering number $\eta_a(G)$.

An elaborate review of results in graphoidal covers with several interesting applications and a large collection of unsolved problems is given in Arumugam et al.[2].

For any graph $G = (V, E)$, $\psi = E$ is trivially an acyclic graphoidal cover and has the interesting property that any two paths in ψ have at most one vertex in common. Motivated by this observation we introduced the concept of simple acyclic graphoidal covers in graphs [3].

Definition 1.7([3]) *A simple acyclic graphoidal cover of a graph G is an acyclic graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of G and is denoted by $\eta_{as}(G)$ or simply η_{as} .*

Definition 1.8 *Let ψ be a collection of internally disjoint paths in G . A vertex of G is said to be an interior vertex of ψ if it is an internal vertex of some path in ψ , otherwise it is said to*

be an exterior vertex of ψ .

Theorem 1.9([3]) *For any simple acyclic graphoidal cover ψ of a graph G , let t_ψ denote the number of exterior vertices of ψ . Let $t = \min t_\psi$, where the minimum is taken over all simple acyclic graphoidal covers ψ of G . Then $\eta_{as}(G) = q - p + t$.*

Theorem 1.10([3]) *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let m denote the number of vertices of degree greater than 2 on C . Then*

$$\eta_{as}(G) = \begin{cases} 3 & \text{if } m = 0 \\ n + 2 & \text{if } m = 1 \\ n + 1 & \text{if } m = 2 \\ n & \text{if } m \geq 3 \end{cases}$$

Theorem 1.11([3]) *Let m and n be integers with $n \geq m \geq 4$. Then*

$$\eta_{as}(K_{m,n}) = \begin{cases} mn - m - n & \text{if } n \leq \binom{m}{2} \\ mn - m - n + r & \text{if } n = \binom{m}{2} + r, r > 0. \end{cases}$$

In this paper we introduce the concept of simple path cover and simple path covering number π_s of a graph G and initiate a study of this parameter. We observe that the concept of simple path cover is a special case of Smarandache path k -cover [8]. For any integer $k \geq 1$, a Smarandache path k -cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ , then a Smarandache path k -cover of G is a simple path cover of G .

§2. Main results

Definition 2.1 *A simple path cover of a graph G is a path cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple path cover of G is called the simple path covering number of G and is denoted by $\pi_s(G)$. Any simple path cover ψ of G for which $|\psi| = \pi_s(G)$ is called a minimum simple path cover of G .*

Example 2.2 Consider the graph G given in Fig.2.1.

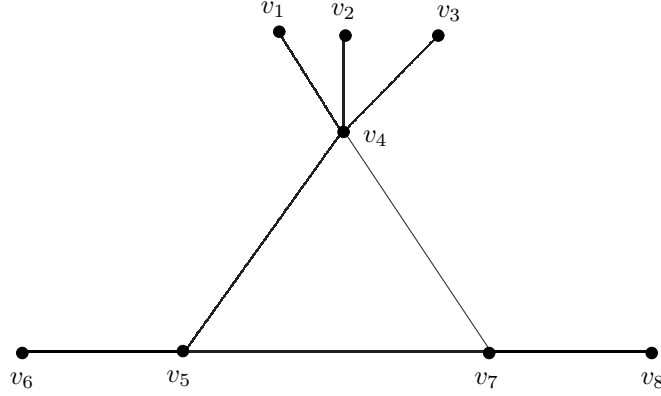


Fig. 2.1

Then $\psi = \{(v_1, v_4, v_7, v_8), (v_3, v_4, v_5, v_6), (v_2, v_4), (v_7, v_5)\}$ is a minimum simple path cover of G so that $\pi_s(G) = 4$.

Remark 2.3 Every path in a simple path cover of a graph G is an induced path.

Theorem 2.4 For any simple path cover ψ of a graph G , let $t_\psi = \sum_{P \in \psi} t(P)$, where $t(P)$ denotes the number of internal vertices of P and let $t = \max t_\psi$, where the maximum is taken over all simple path covers ψ of G . Then $\pi_s(G) = q - t$.

Proof Let ψ be any simple path cover of G . Then

$$\begin{aligned} q &= \sum_{P \in \psi} |E(P)| \\ &= \sum_{P \in \psi} (t(P) + 1) \\ &= |\psi| + \sum_{P \in \psi} t(P) \\ &= |\psi| + t_\psi \end{aligned}$$

Hence $|\psi| = q - t_\psi$ so that $\pi_s(G) = q - t$. \square

Corollary 2.5 For any graph G with k vertices of odd degree $\pi_s(G) = \frac{k}{2} + \sum_{v \in V(G)} \left\lfloor \frac{\deg v}{2} \right\rfloor - t$.

Proof Since $q = \frac{k}{2} + \sum_{v \in V(G)} \left\lfloor \frac{\deg v}{2} \right\rfloor$ the result follows. \square

Corollary 2.6 For any graph G , $\pi_s(G) \geq \frac{k}{2}$ where k is the number of vertices of odd degree in G . Further, the following are equivalent.

- (i) $\pi_s(G) = \frac{k}{2}$.
- (ii) There exists a simple path cover ψ of G such that every vertex v in G is an internal vertex of $\left\lfloor \frac{\deg v}{2} \right\rfloor$ paths in ψ .
- (ii) There exists a simple path cover ψ of G such that every vertex of odd degree is an

external vertex of exactly one path in ψ and no vertex of even degree is an external vertex of any path in ψ .

Remark 2.7 For any (p, q) -graph G , $\pi_s(G) \leq q$. Further, equality holds if and only if G is complete. Hence it follows from Theorem 1.3 that $\pi_s(K_n) = \pi(K_n)$ if and only if $n = 2$.

Remark 2.8 Since any path cover of a tree T is a simple path cover of T , it follows from Theorem 1.2 that $\pi_s(T) = \pi(T) = \frac{k}{2}$, where k is the number of vertices of odd degree in T .

We now proceed to determine the value of π_s for unicyclic graphs and wheels.

Theorem 2.9 Let G be a unicyclic graph with cycle C . Let m denote the number of vertices of degree greater than 2 on C . Let k be the number of vertices of odd degree. Then

$$\pi_s(G) = \begin{cases} 3 & \text{if } m = 0 \\ \frac{k}{2} + 2 & \text{if } m = 1 \\ \frac{k}{2} + 1 & \text{if } m = 2 \\ \frac{k}{2} & \text{if } m \geq 3 \end{cases}$$

Proof Let $C = (v_1, v_2, \dots, v_r, v_1)$.

Case 1. $m = 0$.

Then $G = C$ so that $\pi_s(G) = 3$.

Case 2. $m = 1$.

Let v_1 be the unique vertex of degree greater than 2 on C . Let G_1 be the tree rooted at v_1 . Then G_1 has k vertices of odd degree and hence $\pi_s(G_1) = \frac{k}{2}$. Let ψ_1 be a minimum simple path cover of G_1 .

If $\deg v_1$ is odd, then $\deg_{G_1} v_1$ is odd. Let P be the path in ψ_1 having v_1 as a terminal vertex. Now, let

$$P_1 = P \circ (v_1, v_2)$$

$$P_2 = (v_2, v_3, \dots, v_r) \text{ and}$$

$$P_3 = (v_r, v_1).$$

If $\deg v_1$ is even, then $\deg_{G_1} v_1$ is even. Let $P = (x_1, x_2, \dots, x_r, v_1, x_{r+1}, \dots, x_s)$ be a path in ψ_1 having v_1 as an internal vertex. Now, let

$$P_1 = (x_1, x_2, \dots, x_r, v_1, v_2)$$

$$P_2 = (x_s, x_{s-1}, \dots, x_{r+1}, v_1, v_r) \text{ and}$$

$$P_3 = (v_2, v_3, \dots, v_r).$$

Then $\psi = \{\psi_1 - \{P\}\} \cup \{P_1, P_2, P_3\}$ is a simple path cover of G and hence $\pi_s(G) \leq |\psi_1| + 2 = \frac{k}{2} + 2$. Further, for any simple path cover ψ of G , all the k vertices of odd degree and at least two vertices on C are terminal vertices of paths in ψ . Hence $t \leq q - \frac{k}{2} - 2$, so that $\pi_s(G) = q - t \geq \frac{k}{2} + 2$. Thus $\pi_s(G) = \frac{k}{2} + 2$.

Case 3. $m = 2$.

Let v_1 and v_i , where $2 \leq i \leq r$, be the vertices of degree greater than 2 on C . Let P and Q denote respectively the (v_1, v_i) -section and (v_i, v_1) -section of C . Let v_j be an internal vertex of P (say). Let R_1 and R_2 be the (v_1, v_j) -section of P and (v_j, v_i) -section P respectively. Let G_1 be the graph obtained by deleting all the internal vertices of P .

Subcase 3.1 Both $\deg v_1$ and $\deg v_i$ are odd.

Then both $\deg_{G_1} v_1$ and $\deg_{G_1} v_i$ are even. Hence G_1 is a tree with $k - 2$ odd vertices so that $\pi_s(G_1) = \frac{k}{2} - 1$. Let ψ_1 be a minimum simple path cover of G_1 . Then $\psi = \psi_1 \cup \{R_1, R_2\}$ is a simple path cover of G and $|\psi| = \frac{k}{2} + 1$. Hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Subcase 3.2 Both $\deg v_1$ and $\deg v_i$ are even.

Then $\deg_{G_1} v_1$ and $\deg_{G_1} v_i$ are odd. Hence G_1 is a tree with $k + 2$ vertices of odd degree so that $\pi_s(G_1) = \frac{k}{2} + 1$. Let ψ_1 be a minimum simple path cover of G_1 .

Suppose v_1 and v_i are terminal vertices of two different paths in ψ_1 , say P_1 and P_2 respectively. Now, let

$$\begin{aligned} Q_1 &= P_1 \circ R_1 \\ Q_2 &= P_2 \circ R_2^{-1} \text{ and} \\ \psi &= \{\psi_1 - \{P_1, P_2\}\} \cup \{Q_1, Q_2\}. \end{aligned}$$

Suppose there exists a path P_1 in ψ_1 having both v_1 and v_i as its end vertices. Then let $P_1 = Q$. Let P_2 be an u_1 - w_1 path in ψ_1 having v_1 as an internal vertex and P_3 be an u_2 - w_2 path in ψ_1 having v_i as an internal vertex. Let S_1 and S_2 be the (u_1, v_1) -section of P_2 and (w_1, v_1) -section of P_2 respectively. Let S_3 and S_4 be the (u_2, v_i) -section of P_3 and (w_2, v_i) -section of P_3 respectively. Now, let

$$\begin{aligned} Q_1 &= S_1 \circ P_1 \circ S_3^{-1} \\ Q_2 &= S_2 \circ R_1 \\ Q_3 &= S_4 \circ R_2^{-1} \text{ and} \\ \psi &= \{\psi_1 - \{P_1, P_2, P_3\}\} \cup \{Q_1, Q_2, Q_3\}. \end{aligned}$$

Then ψ is a simple path cover of G and $|\psi| = |\psi_1| = \frac{k}{2} + 1$ and hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Subcase 3.3 $\deg v_1$ is odd and $\deg v_i$ is even.

Then $\deg_{G_1} v_1$ is even and $\deg_{G_1} v_i$ is odd. Hence G_1 is a tree with k vertices of odd degree so that $\pi_s(G_1) = \frac{k}{2}$. Let ψ_1 be a minimum simple path cover of G_1 . Let P_1 be the path in ψ_1 having v_i as a terminal vertex.

If $E(P_1) \cap E(Q) = \phi$, let

$$\begin{aligned} Q_1 &= P_1 \circ R_2^{-1} \\ Q_2 &= R_1 \text{ and} \\ \psi &= \{\psi_1 - \{P_1\}\} \cup \{Q_1, Q_2\}. \end{aligned}$$

Suppose $E(P_1) \cap E(Q) \neq \phi$. Since $\deg_{G_1} v_i \geq 3$, there exists an u_1 - w_1 path in ψ_1 , say P_2 , having v_i as an internal vertex. Let S_1 and S_2 be the (w_1, v_i) -section of P_2 and (u_1, v_i) -section of P_2 respectively. Now, let

$$\begin{aligned} Q_1 &= P_1 \circ S_1^{-1} \\ Q_2 &= S_2 \circ R_2^{-1} \end{aligned}$$

$Q_3 = R_1$ and

$\psi = \{\psi_1 - \{P_1, P_2\}\} \cup \{Q_1, Q_2, Q_3\}$.

Then ψ is a simple path cover of G and $|\psi| = |\psi_1| + 1 = \frac{k}{2} + 1$. Hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Thus in either of the above subcases, we have $\pi_s(G) \leq \frac{k}{2} + 1$. Also, for any simple path cover ψ of G all the k vertices of odd degree and at least one vertex on C are terminal vertices of paths in ψ . Hence $t \leq q - \frac{k}{2} - 1$, so that $\pi_s(G) = q - t \geq \frac{k}{2} + 1$.

Hence $\pi_s(G) = \frac{k}{2} + 1$.

Case 4. $m \geq 3$.

Let $v_{i_1}, v_{i_2}, \dots, v_{i_s}$, where $1 \leq i_1 < i_2 < \dots < i_s \leq r$ and $s \geq 3$, be the vertices of degree greater than 2 on C . Let ψ_{i_j} , $1 \leq j \leq s$, be a minimum simple path cover of the tree rooted at v_{i_j} . Consider the vertices v_{i_1}, v_{i_2} and v_{i_3} . For each j , where $1 \leq j \leq 3$, let P_j be the path in ψ_{i_j} in which v_{i_j} is a terminal vertex if $\deg v_{i_j}$ is odd, otherwise let P_j be an u_j - w_j path in ψ_{i_j} in which v_{i_j} is an internal vertex and R_j and S_j be the (u_j, v_{i_j}) and (w_j, v_{i_j}) -sections of P_j respectively. Further, let $P = (v_{i_1}, v_{i_1+1}, \dots, v_{i_2})$, $Q = (v_{i_2}, v_{i_2+1}, \dots, v_{i_3})$ and $R = (v_{i_3}, v_{i_3+1}, \dots, v_{i_1})$.

If $\deg v_{i_1}, \deg v_{i_2}$ and $\deg v_{i_3}$ are even, let $Q_1 = R_1 \circ P \circ R_2^{-1}$, $Q_2 = S_2 \circ Q \circ R_3^{-1}$ and $Q_3 = S_3 \circ R \circ S_1^{-1}$.

If $\deg v_{i_1}, \deg v_{i_2}$ and $\deg v_{i_3}$ are odd, let $Q_1 = P_1 \circ P$, $Q_2 = P_2 \circ Q$ and $Q_3 = P_3 \circ R$.

If $\deg v_{i_1}, \deg v_{i_2}$ are odd and $\deg v_{i_3}$ is even, let $Q_1 = P_1 \circ P \circ P_2^{-1}$, $Q_2 = R_3 \circ Q^{-1}$ and $Q_3 = S_3 \circ R$.

If $\deg v_{i_1}, \deg v_{i_2}$ are even and $\deg v_{i_3}$ is odd, let $Q_1 = R_1 \circ P \circ R_2^{-1}$, $Q_2 = S_2 \circ Q \circ P_3^{-1}$ and $Q_3 = R \circ S_1^{-1}$.

Then $\psi = (\bigcup_{j=1}^s \psi_{i_j} - \{P_1, P_2, P_3\}) \cup \{Q_1, Q_2, Q_3\}$ is a simple path cover of G such that every vertex of odd degree is an external vertex of exactly one path in ψ and no vertex of even degree is an external vertex of any path in ψ . Hence $\pi_s(G) = \frac{k}{2}$. \square

Corollary 2.10 *Let G be as in Theorem 2.9. Then $\pi_s(G) = \pi(G)$ if and only if $m \geq 3$.*

Proof The proof follows from Theorem 2.9 and Theorem 1.4. \square

We observe that there are infinite families of graphs such as trees and unicyclic graphs having at least three vertices of degree greater than 2 on C for which $\pi_s = \pi$ and so the following problem arises naturally.

Problem 2.11 *Characterize graphs for which $\pi_s = \pi$.*

Theorem 2.12 *For a wheel $W_n = K_1 + C_{n-1}$, we have*

$$\pi_s(W_n) = \begin{cases} 6 & \text{if } n = 4 \\ \lfloor \frac{n}{2} \rfloor + 3 & \text{if } n \geq 5 \end{cases}$$

Proof Let $V(W_n) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ and hence $\pi_s(W_n) = 6$.

Now, suppose $n \geq 5$. Let $r = \lfloor \frac{n}{2} \rfloor$

If n is odd, let

$$\begin{aligned} P_i &= (v_i, v_0, v_{r+i}), i = 1, 2, \dots, r. \\ P_{r+1} &= (v_1, v_2, \dots, v_r), \\ P_{r+2} &= (v_1, v_{2r}, v_{2r-1}, \dots, v_{r+2}) \text{ and} \\ P_{r+3} &= (v_r, v_{r+1}, v_{r+2}). \end{aligned}$$

If n is even, let

$$\begin{aligned} P_i &= (v_i, v_0, v_{r-1+i}), i = 1, 2, \dots, r - 1. \\ P_r &= (v_0, v_{2r-1}), \\ P_{r+1} &= (v_1, v_2, \dots, v_{r-1}), \\ P_{r+2} &= (v_1, v_{2r-1}, \dots, v_{r+1}) \text{ and} \\ P_{r+3} &= (v_{r-1}, v_r, v_{r+1}). \end{aligned}$$

Then $\psi = \{P_1, P_2, \dots, P_{r+3}\}$ is a simple path cover of W_n . Hence $\pi_s(W_n) \leq r+3 = \lfloor \frac{n}{2} \rfloor + 3$. Further, for any simple path cover ψ of W_n at least three vertices on $C = (v_1, v_2, \dots, v_{n-1})$ are terminal vertices of paths in ψ . Hence $t \leq q - \frac{k}{2} - 3$, so that $\pi_s(W_n) = q - t \geq \frac{k}{2} + 3 = \lfloor \frac{n}{2} \rfloor + 3$. Thus $\pi_s(W_n) = \lfloor \frac{n}{2} \rfloor + 3$. \square

Remark 2.13 Since every simple acyclic graphoidal cover of a graph G is a simple path cover of G and every simple path cover of G is a path cover of G , we have $\eta_{as} \geq \pi_s \geq \pi$. These parameters may be either equal or all distinct as shown below. For the graph G_1 given in Figure 2, $\eta_{as}(G_1) = 7, \pi_s(G_1) = 6, \pi(G_1) = 5$ and for the graph G_2 given in Fig.2.2, we have $\eta_{as}(G_2) = \pi_s(G_2) = \pi(G_2) = 3$.

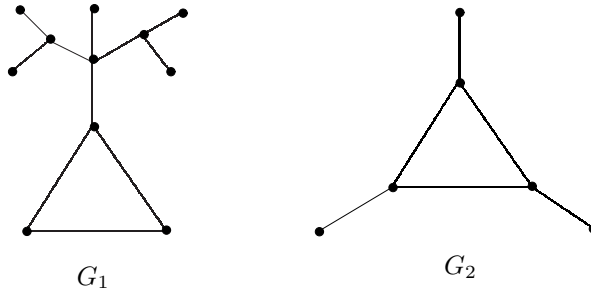


Fig.2.2

Problem 2.14 Characterize graphs for which $\eta_{as} = \pi_s = \pi$.

We now proceed to obtain some bounds for π_s .

Theorem 2.15 For any graph G , $\pi_s(G) \geq \lceil \frac{\Delta}{2} \rceil$. Further, the following are equivalent.

- (i) $\pi_s(G) = \lceil \frac{\Delta}{2} \rceil$.
- (ii) $\eta_{as}(G) = \Delta - 1$.
- (iii) G is homeomorphic to a star.

Proof Since $\pi_s \geq \pi$, the inequality follows from Theorem 1.5.

Suppose $\pi_s(G) = \lceil \frac{\Delta}{2} \rceil$. Let $\psi = \{P_1, P_2, \dots, P_r\}$, where $r = \lceil \frac{\Delta}{2} \rceil$ be a minimum simple path cover of G . Let v be a vertex of G with $deg v = \Delta$. Then v lies on each P_i and v is an internal vertex of all the paths in ψ except possibly for at most one path. Hence $V(P_i) \cap V(P_j) = \{v\}$, for all $i \neq j$, so that G is homeomorphic to a star. Obviously, if G is homeomorphic to a star, then $\pi_s(G) = \lceil \frac{\Delta}{2} \rceil$. Thus (i) and (iii) are equivalent. Similarly the equivalence of (ii) and (iii) can be proved. \square

Theorem 2.16 For any graph G , $\pi_s(G) \geq \binom{\omega}{2}$, where ω is the clique number of G .

Proof Let H be a maximum clique in G so that $|E(H)| = \binom{\omega}{2}$. Let ψ be a simple path cover of G . Since any path in ψ covers at most one edge of H , it follows that $\pi_s(G) \geq \binom{\omega}{2}$. \square

In the following theorem we characterize cubic graphs for which $\pi_s = \binom{\omega}{2}$.

Theorem 2.17 Let G be a cubic graph. Then $\pi_s(G) = \binom{\omega}{2}$ if and only if $G = K_4$.

Proof Let G be a cubic graph with $\pi_s(G) = \binom{\omega}{2}$. Clearly $\omega = 3$ or 4 . Suppose $\omega = 3$. Then it follows from Corollary 2.6 that $\pi_s(G) \geq \frac{p}{2}$ so that $p = 6$. Hence G is isomorphic to the cartesian product $K_3 \times K_2$ and it can be shown that $\pi_s(K_3 \times K_2) = 6 \neq \binom{\omega}{2}$. Thus $\omega = 4$ and consequently $G = K_4$. \square

Problem 2.18 Characterize graphs for which $\pi_s(G) = \binom{\omega}{2}$.

If $\Delta \leq 3$, then every simple path cover of G is a simple acyclic graphoidal cover of G and hence $\eta_{as}(G) = \pi_s(G)$. However, the converse is not true. For the complete graph $K_p (p \geq 5)$, $\pi_s = \eta_{as}$ whereas $\Delta \geq 4$. We now prove that the converse is true for trees and unicyclic graphs.

Theorem 2.19 Let G be a tree. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.

Proof Let G be a tree with $\eta_{as}(G) = \pi_s(G)$.

Suppose $\Delta \geq 4$. Let v be a vertex of G with $deg v \geq 4$.

Let ψ be a minimum simple acyclic graphoidal cover of G . Let P_1 and P_2 be two paths in ψ having v as a terminal vertex. Let $Q = P_1 \circ P_2^{-1}$. Since G is a tree, Q is an induced path and hence $\psi_1 = (\psi - \{P_1, P_2\}) \cup \{Q\}$ is a simple path cover of G with $|\psi_1| = |\psi| - 1 = \eta_{as} - 1$ so that $\pi_s(G) \leq \eta_{as}(G) - 1$, which is a contradiction. Hence $\Delta \leq 3$. \square

Theorem 2.20 Let G be a unicyclic graph. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.

Proof Let G be a unicyclic graph with $\eta_{as}(G) = \pi_s(G)$. Let k denote the number of vertices of odd degree and n be the number of pendant vertices of G .

It follows from Theorem 1.10 and Theorem 2.9 that $k = 2n$. Now, suppose $\Delta > 3$. Then

$$\begin{aligned} 2q &= \sum_{\substack{v \in V(G) \\ deg v = 1}} deg v + \sum_{\substack{v \in V(G) \\ deg v > 1 \\ \text{and is odd}}} deg v + \sum_{\substack{v \in V(G) \\ deg v > 1 \\ \text{and is even}}} deg v \\ &> n + 3(k - n) + 2(p - k) \\ &= 2p, \end{aligned}$$

which is a contradiction. Hence $\Delta \leq 3$. \square

The above results lead to the following problem.

Problem 2.21 Characterize graphs for which $\eta_{as}(G) = \pi_s(G)$.

In the following theorem we establish a relation connecting the parameters η_{as} and π_s .

Theorem 2.22 For any (p, q) -graph G , $\eta_{as}(G) \leq \pi_s(G) + q - p + n - \frac{k}{2}$, where n and k respectively denote the number of pendant vertices and the number of odd vertices of G . Further, equality holds if and only if $\pi_s(G) = \frac{k}{2}$.

Proof Let ψ be a minimum simple path cover of G . Let $i(v)$ denote the number of paths in ψ having $v \in V$ as an internal vertex. If $i(v) \geq 2$, let P_i , where $1 \leq i \leq i(v)$, be the u_i - w_i path in ψ having v as an internal vertex and let Q_i and R_i , where $2 \leq i \leq i(v)$, respectively denote the (v, w_i) -section and (v, u_i) -section of P_i . Let ψ_1 be the collection of paths obtained from ψ by replacing $P_2, P_3, \dots, P_{i(v)}$ by $Q_2, Q_3, \dots, Q_{i(v)}$ and $R_2, R_3, \dots, R_{i(v)}$ for every $v \in V$ with $i(v) \geq 2$. Then ψ_1 is a simple acyclic graphoidal cover of G with $|\psi_1| = \pi_s(G) + \sum_{\substack{v \in V \\ i(v) \geq 2}} (i(v) - 1)$.

Since $i(v) \leq \lfloor \frac{\deg v}{2} \rfloor$, it follows that

$$\begin{aligned}
\eta_{as}(G) &\leq \pi_s(G) + \sum_{\substack{v \in V \\ \deg v \geq 4}} \left(\lfloor \frac{\deg v}{2} \rfloor - 1 \right) \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ \deg v \geq 2}} \left(\lfloor \frac{\deg v}{2} \rfloor - 1 \right) \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ \deg v \geq 2}} \left[\frac{\deg v}{2} \right] - (p - n) \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ \deg v \geq 2 \\ \text{and is odd}}} \frac{\deg v - 1}{2} + \sum_{\substack{v \in V \\ \deg v \geq 2 \\ \text{and is even}}} \frac{\deg v}{2} - p + n \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ \deg v \geq 2 \\ \text{and is odd}}} \frac{\deg v}{2} - \frac{k - n}{2} + \sum_{\substack{v \in V \\ \deg v \geq 2 \\ \text{and is even}}} \frac{\deg v}{2} - p + n \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ \deg v \geq 2}} \frac{\deg v}{2} - \frac{k}{2} + \frac{n}{2} - p + n \\
&= \pi_s(G) + \sum_{v \in V} \frac{\deg v}{2} - \frac{k}{2} - p + n. \\
&= \pi_s(G) + q - p + n - \frac{k}{2}.
\end{aligned}$$

Thus $\eta_{as}(G) \leq \pi_s(G) + q - p + n - \frac{k}{2}$. Further, it is clear that $\eta_{as}(G) = \pi_s(G) + q - p + n - \frac{k}{2}$ if and only if there exist a minimum simple path cover ψ of G such that $i(v) = \lfloor \frac{\deg v}{2} \rfloor$ for all $v \in V$. Hence it follows from Corollary 2.6 that $\eta_{as}(G) = \pi_s(G) + q - p + n - \frac{k}{2}$ if and only if $\pi_s = \frac{k}{2}$. \square

Corollary 2.23 If $\pi_s(G) = \frac{k}{2}$, then $\eta_{as}(G) = q - p + n$.

Proof Suppose $\pi_s(G) = \frac{k}{2}$. By Theorem 2.22, we have $\eta_{as}(G) \leq q - p + n$. Hence it follows from Theorem 1.9 that $\eta_{as}(G) = q - p + n$. \square

Remark 2.24 The converse of Corollary 2.23 is not true. For example, $\eta_{as}(K_{4,5}) = q - p = 11$, whereas $\pi_s(K_{4,5}) > 2 = \frac{k}{2}$.

References

- [1] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure appl. Math.*, **18**(10)(1987), 882 - 890.
- [2] S. Arumugam, B. D. Acharya and E. Sampathkumar, Graphoidal covers of a graph - A creative review, *Proceedings of the National workshop on Graph Theory and its Applications*, Manonmaniam Sundaranar University, Tirunelveli, Eds. S. Arumugam, B. D. Acharya and E. Sampathkumar, Tata McGraw Hill, (1996), 1 - 28.
- [3] S. Arumugam and I. Sahul Hamid, Simple Acyclic Graphoidal covers in a graph, *Australasian Journal of Combinatorics*, **37**(2007), 243 -255.
- [4] S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Math.*, **190**(1998),67 - 77.
- [5] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass, 1972.
- [6] F. Harary, Covering and packing in graphs I, *Ann. N. Y. Acad. Sci.*, **175**(1970), 198 - 205.
- [7] F. Harary and A. J. Schwenk, Evolution of the path number of a graph, covering and packing in graphs II, *Graph Theory and Computing*, Ed. R. C. Road, Academic Press, New York, (1972), 39 - 45.
- [8] L.F. Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005
- [9] B. Peroche, The path number of some multipartite graphs, *Annals of Discrete Math.*, **9**(1982), 193 - 197.
- [10] R. G. Stanton, D. D. Cowan and L. O. James, Some results on path numbers, *Proc. Louisiana Conf. on Combinatorics, Graph Theory and computing* (1970), 112 - 135.
- [11] R. G. Stanton, D. D. Cowan and L. O. James, Tripartite path number, *Graph Theory and Computing*, Eds. R. C. Read, Academic Press, New York, (1973), 285 - 294.