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Singed Total Domatic Number of a Graph

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Abstract: Let G be a finite and simple graph with vertex set $V(G)$, $k \geq 1$ an integer and let $f : V(G) \to \{-k, k-1, \cdots, -1, 1, \cdots, k-1, k\}$ be $2k$ valued function. If $\sum\limits_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a Smarandachely k-Signed total dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of Smarandachely k-Signed total dominating function on G with the property that $\sum\limits_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(G)$ is called a Smarandachely k-Signed total dominating family (function) on G. Particularly, a Smarandachely 1-Signed total dominating function or family is called signed total dominating function or family on G. The maximum number of functions in a signed total dominating family on G is the signed total domatic number of G. In this paper, some properties related signed total domatic number and signed total domination number of a graph are studied and found the sign total domatic number of certain class of graphs such as fans, wheels and generalized Petersen graph.

Key Words: Smarandachely k-signed total dominating function, signed total domination number, signed total domatic number.

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§1. Terminology and Introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants [1] and [4]. We considered finite, undirected, simple graphs $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The order of G is given by $n = |V(G)|$. If $v \in V(G)$, then the open neighborhood of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. The number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex $v \in V(G)$, and $\delta(G)$ is the *minimum degree* of G. The *complete graph* and the *cycle* of order n are denoted by K_n and C_n respectively. A fan and a wheel is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all the vertices of the path and cycle respectively. The generalized Petersen graph $P(n, k)$ is defined to be a graph on $2n$ vertices with $V(P(n, k)) = \{v_i u_i : 1 \leq i \leq n\}$ and $E(P(n, k)) =$

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$\{v_iv_{i+1}, v_iu_i, u_iu_{i+k} : 1 \leq i \leq n$, subscripts modulo $n\}$. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ in to some set of numbers, then $f(A) = \sum_{x\in A} f(x)$.

Let $k \geq 1$ be an integer and let $f : V(G) \to \{-k, k-1, \cdots, -1, 1, \cdots, k-1, k\}$ be $2k$ valued function. If $\sum\limits_{x\in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a Smarandachely k-Signed total dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of Smarandachely k-Signed total dominating function on G with the property that $\sum\limits_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(G)$ is called a Smarandachely k-Signed total dominating family (function) on G. Particularly, a Smarandachely 1-Signed total dominating function or family is called signed total dominating function or family on G. The singed total dominating function is defined in [6] as a two valued function $f : V(G) \to \{-1, 1\}$ such that $\sum_{x\in N(v)} f(x) \geq 1$ for each $v \in V(G)$. The minimum of weights $w(f)$, taken over all signed total dominating functions f on G, is called the signed total domination number $\gamma_t^s(G)$. Signed total domination has been studied in [3].

A set $\{f_1, f_2, \ldots, f_d\}$ of signed total dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each $x \in V(G)$, is called a signed total dominating family on G. The maximum number of functions in a signed total dominating family is the signed total domatic number of G, denoted by $d_t^s(G)$. Signed total domatic number was introduced by Guan Mei and Shan Er-fang [2]. Guan Mei and Shan Er-fang [2] have determined the basic properties of $d_t^s(G)$. Some of them are analogous to those of the signed domatic number in [5] and studied sharp bounds of the signed total domatic number of regular graphs, complete bipartite graphs and complete graphs. Guan Mei and Shan Er-fang [2] presented the following results which are useful in our investigations.

Proposition 1.1([6]) *For Circuit C_n of length n we have $\gamma_t^s(C_n) = n$.*

Proof Here no other signed total dominating exists than the constants equal to 1. □

Theorem 1.2([3]) *Let T be a tree of order $n \geq 2$. then, $\gamma_t^s(T) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2.*

Proposition 1.3([2]) *The signed total domatic number $d_t^s(G)$ is well defined for each graph G.*

Proposition 1.4([2]) *For any graph G of order n ,$\gamma_t^s(G) \cdot d_t^s(G) \leq n$.*

Proposition 1.5([2]) *If G is a graph with the minimum degree $\delta(G)$, then $1 \leq d_t^s(G) \leq \delta(G)$.*

Proposition 1.6([2]) *The signed total domatic number is an odd integer.*

Corollary 1.7([2]) *If G is a graph with the minimum degree $\delta(G) = 1$ or 2, then $d_t^s(G) = 1$. In particular, $d_t^s(C_n) = d_t^s(P_n) = d_t^s(K_{1,n-1}) = d_t^s(T) = 1$, where T is a tree.*

§2. Properties of the Signed Total Domatic Number

Proposition 2.1 *If G is a graph of order n and $\gamma_t^s(G) \geq 0$ then, $\gamma_t^s(G) + d_t^s(G) \leq n+1$ equality* *holds if and only if G is isomorphic to C_n or tree T of order $n \geq 2$.*

Proof Let G be a graph of order n. The inequality follows from the fact that for any two non-negative integers a and b, $a + b \leq ab + 1$. By Proposition 1.4 we have,

$$\gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$$

Suppose that $\gamma_t^s(G) + d_t^s(G) = n + 1$ then, $n + 1 = \gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$.

This implies that $\gamma_t^s(G) + d_t^s(G) = \gamma_t^s(G) \cdot d_t^s(G) + 1$. This shows that $\gamma_t^s(G) \cdot d_t^s(G) = n$ Solving equations 1 and 2 simultaneously, we have either $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ or $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$. If $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ then $n = d_t^s(G) \leq \delta(G)$ There fore, $\delta(G) \geq n$ a contradiction.

If $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$ then by Proposition 1.1 and Proposition 1.2, we have $\gamma_t^s(C_n) = n$ and $d_t^s(C_n) = 1$ and By Theorem 1.2, If T is a tree of order $n \geq 2$ then, $\gamma_t^s(T) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2 and $d_t^s(T) = 1$. □

Theorem 2.2 *Let G be a graph of order n then $d_t^s(G) + d_t^s(\bar{G}) \leq n - 1$.*

Proof Let G be a regular graph order n, By Proposition 1.5 we have $d_t^s(G) \leq \delta(G)$ and $d_t^s(\bar{G}) \leq \delta(\bar{G})$. Thus we have,

$$d_t^s(G) + d_t^s(\bar{G}) \leq \delta(G) + \delta(\bar{G}) = \delta(G) + (n - 1 - \Delta(G)) \leq n - 1.$$

Thus the inequality holds. □

§3. Signed Total Domatic Number of Fans, Wheels and Generalized Petersen Graph

Proposition 3.1 *Let G be a fan of order n then $d_t^s(G) = 1$.*

Proof Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be the vertex set of the fan G such that $x_1, x_2, \ldots, x_n, x_1$ is a cycle of length n and x_n is adjacent to x_i for each $i = 2, 3, \ldots, n-2$. By Proposition 1.5 and Proposition 1.6, $1 \leq d_t^s(G) \leq \delta(G) = 2$, which implies $d_t^s(G) = 1$ which proves the result.□

Proposition 3.2 *If G is a wheel of order n then $d_t^s(G) = 1$.*

Proof Let $x_1, x_2, \ldots, x_n$ be the vertex set of the wheel G such that $x_1, x_2, \ldots, x_{n-1}, x_1$ is a cycle of length $n - 1$ and x_n is adjacent to x_i for each $i = 1, 2, 3, \ldots, n-1$. According to the Proposition 1.5 and Proposition 1.6, we observe that either $d_t^s(G) = 1$ or $d_t^s(G) = 3$. Suppose to the contrary that $d_t^s(G) = 3$. Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating family. Because of $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$, there exists at least one function say f_1 with $f_1(x_n) = -1$ The condition $\sum_{x \in N(v)} f_1(x) \geq 1$ for each $v \in (V(G) - \{x_n\})$ yields $f_1(x) = 1$ for each some $i \in \{1, 2, \ldots, n-1\}$ and $t = 2, 3$ then it follows that $f_t(x_{i+1}) = f_t(x_{i+2}) = 1$, where the indices are taken taken modulo $n - 1$ and $f_t(x_n) = 1$. Consequently, the function f_t has at most $\lfloor \frac{n}{2} \rfloor - 1$ for n is odd and $\frac{n}{2} - 1$ for n is even number of vertices $x \in V(G)$ such that

$f_t(x) = -1$. Thus there exist at most $\lfloor \frac{n}{2} \rfloor - 1$ for n is odd and $\frac{n}{2} - 1$ for n is even number of vertices $x \in V(G)$ such that $f_t(x) = -1$ for at least one $i = 1, 2, 3$. Since $n \geq 4$, we observe that $2(\lfloor \frac{n}{2} \rfloor + 1) = 2\left(\frac{n}{2} - 1\right) + 1 < n$ for n is odd and $2\left(\frac{n}{2} - 1\right) + 1 < n$, a contradiction to $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$ for each $x \in V(G)$. □

Proposition 3.3 *Let $G = P(n,k)$ be a generalized Petersen graph then for $k = 1, 2$, $d_t^s(G) = 1$.*

Proof The generalized Petersen graph $P(n,1)$ is a graph on $2n$ vertices with

$$V(P(n,k)) = \{v_i u_i : 1 \leq i \leq n\}$$

and $E(P(n,k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+1} : 1 \leq i \leq n$, subscripts modulo $n\}$. According to the Proposition 1.5, Proposition 1.6, we observe that $d_t^s(G) = 1$ or $d_t^s(G) = 3$.

Case 1: $k = 1$

Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating functions. Because of $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $i \in \{1, 2, \ldots, 2n\}$, there exist at least one number $j \in \{1, 2, 3\}$ such that $f_j(v_i) = -1$. Let, for example, $f_1(v_k) = -1$ for for any $t \in \{1, 2, \ldots, 2n\}$ then $\sum_{x \in N(v_t)} f_1(v) \geq 1$ implies that $f_1(v_k) = f_1(v_{k+1}) = -1$ for $k \cong 0, 1 mod 4$ and $f_1(v_k) = -1$ for $k \cong 0 mod 3$. This implies, there exist at most $8r$, $8r + 2$, $8r + 4$, $8r + 6$, $r \geq 1$ vertices such that $f_t(v) = -1$ for each $t = 2, 3$ when $P(n,1)$ is of order $2(6r + l)$ for $0 \leq l \leq 2$, $2(6r + 3)$, $2(6r + 4)$, $2(6r + 5)$ respectively. Thus there exist $3(8r) = 3(8(\frac{n}{12} - \frac{l}{6}) < n$ (similarly $< n$ for all values of vertex set) a contradiction to $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $v \in V(G)$.

Case 2: $k = 2$

Similar to the proof of Case 1, we can prove the claim in this case. □

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