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# Smarandache isotopy theory of Smarandache: quasigroups and loops

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**Abstract** The concept of Smarandache isotopy is introduced and its study is explored for Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops. The exploration includes: Smarandache; isotopy and isomorphy classes, Smarandache $f, g$ principal isotopes and G-Smarandache loops.

**Keywords** Smarandache, groupoids, quasigroups, loops, $f$, $g$ principal isotopes.

## §1. Introduction

In 2002, W. B. Vasantha Kandasamy initiated the study of Smarandache loops in her book [12] where she introduced over 75 Smarandache concepts in loops. In her paper [13], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. For more on loops and their properties, readers should check[11], [1], [3], [4],[5] and [12]. In [12], Page 102, the author introduced Smarandache isotopes of loops particularly Smarandache principal isotopes. She has also introduced the Smarandache concept in some other algebraic structures as [14][15][16][17][18][19] account. The present author has contributed to the study of S-quasigroups and S-loops in [6], [7] and [8] while Muktibodh [10] did a study on the first.

In this study, the concept of Smarandache isotopy will be introduced and its study will be explored in Smarandache: groupoids, quasigroups and loops just like the study of isotopy theory was carried out for groupoids, quasigroups and loops as summarized in Bruck [1], Dene and Keedwell [4], Pflugfelder [11].

## §2. Definitions and notations

**Definition 2.1.** Let $L$ be a non-empty set. Define a binary operation $(\cdot)$ on $L$ : If $x \cdot y \in L \;\forall\; x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ; $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for $x$ and $y$ respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that $\forall\; x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

If there exists at least a non-empty and non-trivial subset $M$ of a groupoid(quasigroup or semigroup or loop) $L$ such that $(M, \cdot)$ is a non-trivial subsemigroup(subgroup or subgroup or subgroup) of $(L,\cdot)$, then $L$ is called a Smarandache: groupoid(S-groupoid)$\Big($quasigroup(S-quasigroup) or semigroup(S-semigroup) or loop(S-loop)$\Big)$ with Smarandache: subsemigroup(S-subsemigroup)$\Big($subgroup(S-subgroup) or subgroup(S-subgroup) or subgroup(S-subgroup)$\Big)$ $M$.

Let $(G,\cdot)$ be a quasigroup(loop). The bijection $L_x : G \to G$ defined as $yL_x = x\cdot y \; \forall \; x,y \in G$ is called a left translation(multiplication) of $G$ while the bijection $R_x : G \to G$ defined as $yR_x = y \cdot x \; \forall \; x,y \in G$ is called a right translation(multiplication) of $G$.

The set $SYM(L,\cdot) = SYM(L)$ of all bijections in a groupoid $(L,\cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(L,\cdot)$.

**Definition 2.2.** If $(L,\cdot)$ and $(G,\circ)$ are two distinct groupoids, then the triple $(U,V,W) : (L,\cdot) \to (G,\circ)$ such that $U,V,W : L \to G$ are bijections is called an isotopism if and only if

$$xU \circ yV = (x \cdot y)W \; \forall \; x,y \in L.$$

So we call $L$ and $G$ groupoid isotopes. If $L = G$ and $W = I$(identity mapping) then $(U,V,I)$ is called a principal isotopism, so we call $G$ a principal isotope of $L$. But if in addition $G$ is a quasigroup such that for some $f,g \in G$, $U = R_g$ and $V = L_f$, then $(R_g, L_f, I) : (G,\cdot) \to (G,\circ)$ is called an $f,g$-principal isotopism while $(G,\cdot)$ and $(G,\circ)$ are called quasigroup isotopes.

If $U = V = W$, then $U$ is called an isomorphism, hence we write $(L,\cdot) \cong (G,\circ)$. A loop $(L,\cdot)$ is called a G-loop if and only if $(L,\cdot) \cong (G,\circ)$ for all loop isotopes $(G,\circ)$ of $(L,\cdot)$.

Now, if $(L,\cdot)$ and $(G,\circ)$ are S-groupoids with S-subsemigroups $L'$ and $G'$ respectively such that $(G')A = L'$, where $A \in \{U,V,W\}$, then the isotopism $(U,V,W) : (L,\cdot) \to (G,\circ)$ is called a Smarandache isotopism(S-isotopism). Consequently, if $W = I$ the triple $(U,V,I)$ is called a Smarandache principal isotopism. But if in addition $G$ is a S-quasigroup with S-subgroup $H'$ such that for some $f,g \in H$, $U = R_g$ and $V = L_f$, and $(R_g, L_f, I) : (G,\cdot) \to (G,\circ)$ is an isotopism, then the triple is called a Smarandache $f,g$-principal isotopism while $f$ and $g$ are called Smarandache elements(S-elements).

Thus, if $U = V = W$, then $U$ is called a Smarandache isomorphism, hence we write $(L,\cdot) \succsim (G,\circ)$. An S-loop $(L,\cdot)$ is called a G-Smarandache loop(GS-loop) if and only if $(L,\cdot) \succsim (G,\circ)$ for all loop isotopes(or particularly all S-loop isotopes) $(G,\circ)$ of $(L,\cdot)$.

**Example 2.1.** The systems $(L,\cdot)$ and $(L,*)$, $L = \{0,1,2,3,4\}$ with the multiplication tables below are S-quasigroups with S-subgroups $(L',\cdot)$ and $(L'',*)$ respectively, $L' = \{0,1\}$ and $L'' = \{1,2\}$. $(L,\cdot)$ is taken from Example 2.2 of [10]. The triple $(U,V,W)$ such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}, \; V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 0 & 3 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 4 & 3 \end{pmatrix}$$

are permutations on $L$, is an S-isotopism of $(L,\cdot)$ onto $(L,*)$. Notice that $A(L') = L''$ for all $A \in \{U,V,W\}$ and $U,V,W : L' \to L''$ are all bijcetions.

| · | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 3 | 4 | 2 |
| 1 | 1 | 0 | 2 | 3 | 4 |
| 2 | 3 | 4 | 1 | 2 | 0 |
| 3 | 4 | 2 | 0 | 1 | 3 |
| 4 | 2 | 3 | 4 | 0 | 1 |

| ∗ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 4 | 2 | 3 |
| 1 | 3 | 1 | 2 | 0 | 4 |
| 2 | 4 | 2 | 1 | 3 | 0 |
| 3 | 0 | 4 | 3 | 1 | 2 |
| 4 | 2 | 3 | 0 | 4 | 1 |

**Example 2.2.** According to Example 4.2.2 of [15], the system $(\mathbb{Z}_6, \times_6)$ i.e the set $L = \mathbb{Z}_6$ under multiplication modulo 6 is an S-semigroup with S-subgroups $(L', \times_6)$ and $(L'', \times_6)$, $L' = \{2, 4\}$ and $L'' = \{1, 5\}$. This can be deduced from its multiplication table, below. The triple $(U, V, W)$ such that

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 & 0 \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 4 & 2 & 3 \end{pmatrix}$$

are permutations on $L$, is an S-isotopism of $(\mathbb{Z}_6, \times_6)$ unto an S-semigroup $(\mathbb{Z}_6, *)$ with S-subgroups $(L''', *)$ and $(L'''', *)$, $L''' = \{2, 5\}$ and $L'''' = \{0, 3\}$ as shown in the second table below. Notice that $A(L') = L'''$ and $A(L'') = L''''$ for all $A \in \{U, V, W\}$ and $U, V, W : L' \to L'''$ and $U, V, W : L'' \to L''''$ are all bijcetions.

| $\times_6$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 |
| 5 | 0 | 5 | 4 | 3 | 2 | 1 |

| ∗ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 4 | 1 | 1 | 4 | 4 | 1 |
| 2 | 5 | 1 | 5 | 2 | 1 | 2 |
| 3 | 3 | 1 | 5 | 0 | 4 | 2 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 2 | 1 | 2 | 5 | 1 | 5 |

**Remark 2.1.** Taking careful look at Definition 2.2 and comparing it with Definition 4.4.1[12], it will be observed that the author did not allow the component bijections $U$,$V$ and $W$ in $(U, V, W)$ to act on the whole S-loop $L$ but only on the S-subloop(S-subgroup) $L'$. We feel this is necessary to adjust here so that the set $L - L'$ is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [12] only take care of the structure of the elements in the S-subloop and not the S-loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.

## §3. Smarandache Isotopy and Isomorphy classes

**Theorem 3.1.** Let $\mathfrak{G} = \Big\{ \big(G_\omega, \circ_\omega\big) \Big\}_{\omega\in\Omega}$ be a set of distinct S-groupoids with a corresponding set of S-subsemigroups $\mathfrak{H} = \Big\{ \big(H_\omega, \circ_\omega\big) \Big\}_{\omega\in\Omega}$. Define a relation $\sim$ on $\mathfrak{G}$ such that for all $\big(G_{\omega_i}, \circ_{\omega_i}\big)$ , $\big(G_{\omega_j}, \circ_{\omega_j}\big) \in \mathfrak{G}$, where $\omega_i, \omega_j \in \Omega$,

$$\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_j}, \circ_{\omega_j}\big) \Longleftrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big) \text{ and } \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ are S-isotopic.}$$

Then $\sim$ is an equivalence relation on $\mathfrak{G}$.

**Proof.** Let $\big(G_{\omega_i}, \circ_{\omega_i}\big)$ , $\big(G_{\omega_j}, \circ_{\omega_j}\big)$, $\big(G_{\omega_k}, \circ_{\omega_k}\big), \in \mathfrak{G}$, where $\omega_i, \omega_j, \omega_k \in \Omega$.

**Reflexivity** If $I \ : \ G_{\omega_i} \to G_{\omega_i}$ is the identity mapping, then

$$xI \circ_{\omega_i} yI = (x \circ_{\omega_i} y)I \ \forall \ x, y \in G_{\omega_i} \Longrightarrow \text{ the triple } (I, I, I) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \to \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

is an S-isotopism since $\big(H_{\omega_i}\big)I = H_{\omega_i} \ \forall \ \omega_i \in \Omega$. In fact, it can be simply deduced that every S-groupoid is S-isomorphic to itself.

**Symmetry** Let $\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_j}, \circ_{\omega_j}\big)$. Then there exist bijections

$$U, V, W \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ such that } \big(H_{\omega_i}\big)A = H_{\omega_j} \ \forall \ A \in \{U, V, W\}$$

so that the triple

$$\alpha = (U, V, W) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big)$$

is an isotopism. Since each of $U, V, W$ is bijective, then their inverses

$$U^{-1}, V^{-1}, W^{-1} \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

are bijective. In fact, $\big(H_{\omega_j}\big)A^{-1} = H_{\omega_i} \ \forall \ A \in \{U, V, W\}$ since $A$ is bijective so that the triple

$$\alpha^{-1} = (U^{-1}, V^{-1}, W^{-1}) \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_i}, \circ_{\omega_i}\big)$$

is an isotopism. Thus, $\big(G_{\omega_j}, \circ_{\omega_j}\big) \sim \big(G_{\omega_i}, \circ_{\omega_i}\big)$.

**Transitivity** Let $\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_j}, \circ_{\omega_j}\big)$ and $\big(G_{\omega_j}, \circ_{\omega_j}\big) \sim \big(G_{\omega_k}, \circ_{\omega_k}\big)$. Then there exist bijections

$$U_1, V_1, W_1 \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ and } U_2, V_2, W_2 \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

$$\text{such that } \big(H_{\omega_i}\big)A = H_{\omega_j} \ \forall \ A \in \{U_1, V_1, W_1\}$$

$$\text{and } \big(H_{\omega_j}\big)B = H_{\omega_k} \ \forall \ B \in \{U_2, V_2, W_2\} \text{ so that the triples}$$

$$\alpha_1 = (U_1, V_1, W_1) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_j}, \circ_{\omega_j}\big) \text{ and}$$

$$\alpha_2 = (U_2, V_2, W_2) \ : \ \big(G_{\omega_j}, \circ_{\omega_j}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

are isotopisms. Since each of $U_i, V_i, W_i$, $i = 1, 2$, is bijective, then

$$U_3 = U_1U_2, V_3 = V_1V_2, W_3 = W_1W_2 \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

are bijections such that $\big(H_{\omega_i}\big)A_3 = \big(H_{\omega_i}\big)A_1A_2 = \big(H_{\omega_j}\big)A_2 = H_{\omega_k}$ so that the triple

$$\alpha_3 = \alpha_1\alpha_2 = (U_3, V_3, W_3) \ : \ \big(G_{\omega_i}, \circ_{\omega_i}\big) \longrightarrow \big(G_{\omega_k}, \circ_{\omega_k}\big)$$

is an isotopism. Thus, $\big(G_{\omega_i}, \circ_{\omega_i}\big) \sim \big(G_{\omega_k}, \circ_{\omega_k}\big)$.

**Remark 3.1.** As a follow up to Theorem 3.1, the elements of the set $\mathfrak{G}/\sim$ will be referred to as Smarandache isotopy classes(S-isotopy classes). Similarly, if $\sim$ meant "S-isomorphism" in Theorem 3.1, then the elements of $\mathfrak{G}/\sim$ will be referred to as Smarandache isomorphy classes(S-isomorphy classes). Just like isotopy has an advantage over isomorphy in the classification of loops, so also S-isotopy will have advantage over S-isomorphy in the classification of S-loops.

**Corollary 3.1.** Let $\mathcal{L}_n$, $\mathcal{SL}_n$ and $\mathcal{NSL}_n$ be the sets of; all finite loops of order $n$; all finite S-loops of order $n$ and all finite non S-loops of order $n$ respectively.

1. If $\mathcal{A}_i^n$ and $\mathcal{B}_i^n$ represent the isomorphy class of $\mathcal{L}_n$ and the S-isomorphy class of $\mathcal{SL}_n$ respectively, then

(a) $|\mathcal{SL}_n| + |\mathcal{NSL}_n| = |\mathcal{L}_n|$;

(i) $|\mathcal{SL}_5| + |\mathcal{NSL}_5| = 56$,

(ii) $|\mathcal{SL}_6| + |\mathcal{NSL}_6| = 9,408$

(iii) $|\mathcal{SL}_7| + |\mathcal{NSL}_7| = 16,942,080$.

(b) $|\mathcal{NSL}_n| = \sum\limits_{i=1} |\mathcal{A}_i^n| - \sum\limits_{i=1} |\mathcal{B}_i^n|$;

(i) $|\mathcal{NSL}_5| = \sum\limits_{i=1}^{6} |\mathcal{A}_i^5| - \sum\limits_{i=1} |\mathcal{B}_i^5|$,

(ii) $|\mathcal{NSL}_6| = \sum\limits_{i=1}^{109} |\mathcal{A}_i^6| - \sum\limits_{i=1} |\mathcal{B}_i^6|$

(iii) $|\mathcal{NSL}_7| = \sum\limits_{i=1}^{23,746} |\mathcal{A}_i^7| - \sum\limits_{i=1} |\mathcal{B}_i^7|$.

2. If $\mathfrak{A}_i^n$ and $\mathfrak{B}_i^n$ represent the isotopy class of $\mathcal{L}_n$ and the S-isotopy class of $\mathcal{SL}_n$ respectively, then

$$|\mathcal{NSL}_n| = \sum_{i=1} |\mathfrak{A}_i^n| - \sum_{i=1} |\mathfrak{B}_i^n|;$$

(i) $|\mathcal{NSL}_5| = \sum\limits_{i=1}^{2} |\mathfrak{A}_i^5| - \sum\limits_{i=1} |\mathfrak{B}_i^5|$,

(ii) $|\mathcal{NSL}_6| = \sum\limits_{i=1}^{22} |\mathfrak{A}_i^6| - \sum\limits_{i=1} |\mathfrak{B}_i^6|$ and

(iii) $|\mathcal{NSL}_7| = \sum\limits_{i=1}^{564} |\mathfrak{A}_i^7| - \sum\limits_{i=1} |\mathfrak{B}_i^7|$.

**Proof.** An S-loop is an S-groupoid. Thus by Theorem 3.1, we have S-isomorphy classes and S-isotopy classes. Recall that $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n| - |\mathcal{SL}_n \bigcap \mathcal{NSL}_n|$ but $\mathcal{SL}_n \bigcap \mathcal{NSL}_n = \emptyset$ so $|\mathcal{L}_n| = |\mathcal{SL}_n| + |\mathcal{NSL}_n|$. As stated and shown in [11], [15], [2] and [9], the facts in Table 1 are true where $n$ is the order of a finite loop. Hence the claims follow.

**Question 3.1.** How many S-loops are in the family $\mathcal{L}_n$? That is, what is $|\mathcal{SL}_n|$ or $|\mathcal{NSL}_n|$.

**Theorem 3.2.** Let $(G, \cdot)$ be a finite S-groupoid of order $n$ with a finite S-subsemigroup $(H, \cdot)$ of order $m$. Also, let

$$\mathcal{ISOT}(G, \cdot),\ \mathcal{SISOT}(G, \cdot) \text{ and } \mathcal{NSISOT}(G, \cdot)$$

be the sets of all isotopisms, S-isotopisms and non S-isotopisms of $(G, \cdot)$. Then,

$$\mathcal{ISOT}(G, \cdot) \text{ is a group and } \mathcal{SISOT}(G, \cdot) \leq \mathcal{ISOT}(G, \cdot).$$

Furthermore:

1. $|\mathcal{ISOT}(G, \cdot)| = (n!)^3$;
2. $|\mathcal{SISOT}(G, \cdot)| = (m!)^3$;
3. $|\mathcal{NSISOT}(G, \cdot)| = (n!)^3 - (m!)^3$.

**Proof.**

1. This has been shown to be true in [Theorem 4.1.1, [4]].
2. An S-isotopism is an isotopism. So, $\mathcal{SISOT}(G, \cdot) \subset \mathcal{ISOT}(G, \cdot)$. Thus, we need to just verify the axioms of a group to show that $\mathcal{SISOT}(G, \cdot) \leq \mathcal{ISOT}(G, \cdot)$. These can be done using the proofs of reflexivity, symmetry and transitivity in Theorem 3.1 as guides. For all triples

$$\alpha \in \mathcal{SISOT}(G, \cdot) \text{ such that } \alpha = (U, V, W)\ :\ (G, \cdot) \longrightarrow (G, \circ),$$

where $(G, \cdot)$ and $(G, \circ)$ are S-groupoids with S-subgroups $(H, \cdot)$ and $(K, \circ)$ respectively, we can set

$$U' := U|_H,\ V' := V|_H \text{ and } W' := W|_H \text{ since } A(H) = K\ \forall\ A \in \{U, V, W\},$$

so that $\mathcal{SISOT}(H, \cdot) = \{(U', V', W')\}$. This is possible because of the following arguments.

Let

$$X = \Big\{f' := f|_H\ \Big|\ f : G \longrightarrow G,\ f : H \longrightarrow K \text{ is bijective and } f(H) = K\Big\}.$$

Let

$$SYM(H, K) = \{\text{bijections from } H \text{ unto } K\}.$$

| $n$ | 5 | 6 | 7 |
|---|---|---|---|
| $\vert\mathcal{L}_n\vert$ | 56 | 9, 408 | 16, 942, 080 |
| $\{\mathcal{A}_i^n\}_{i=1}^k$ | $k = 6$ | $k = 109$ | $k = 23, 746$ |
| $\{\mathfrak{A}_i^n\}_{i=1}^m$ | $m = 2$ | $m = 22$ | $m = 564$ |

Table 1: Enumeration of Isomorphy and Isotopy classes of finite loops of small order

By definition, it is easy to see that $X \subseteq SYM(H,K)$. Now, for all $U \in SYM(H,K)$, define $U : H^c \longrightarrow K^c$ so that $U : G \longrightarrow G$ is a bijection since $|H| = |K|$ implies $|H^c| = |K^c|$. Thus, $SYM(H,K) \subseteq X$ so that $SYM(H,K) = X$.

Given that $|H| = m$, then it follows from (1) that

$$|\mathcal{ISOT}(H,\cdot)| = (m!)^3 \text{ so that } |\mathcal{SISOT}(G,\cdot)| = (m!)^3 \text{ since } SYM(H,K) = X.$$

3.

$$\mathcal{NSISOT}(G,\cdot) = \big(\mathcal{SISOT}(G,\cdot)\big)^c.$$

So, the identity isotopism

$$(I,I,I) \notin \mathcal{NSISOT}(G,\cdot), \text{ hence } \mathcal{NSISOT}(G,\cdot) \nleq \mathcal{ISOT}(G,\cdot).$$

Furthermore,

$$|\mathcal{NSISOT}(G,\cdot)| = (n!)^3 - (m!)^3.$$

**Corollary 3.2.** Let $(G,\cdot)$ be a finite S-groupoid of order $n$ with an S-subsemigroup $(H,\cdot)$. If $\mathcal{ISOT}(G,\cdot)$ is the group of all isotopisms of $(G,\cdot)$ and $S_n$ is the symmetric group of degree $n$, then

$$\mathcal{ISOT}(G,\cdot) \succsim S_n \times S_n \times S_n.$$

**Proof.** As concluded in [Corollary 1, [4]], $\mathcal{ISOT}(G,\cdot) \cong S_n \times S_n \times S_n$. Let $\mathcal{PISOT}(G,\cdot)$ be the set of all principal isotopisms on $(G,\cdot)$. $\mathcal{PISOT}(G,\cdot)$ is an S-subgroup in $\mathcal{ISOT}(G,\cdot)$ while $S_n \times S_n \times \{I\}$ is an S-subgroup in $S_n \times S_n \times S_n$. If

$$\Upsilon : \mathcal{ISOT}(G,\cdot) \longrightarrow S_n \times S_n \times S_n \text{ is defined as}$$

$$\Upsilon\big((A,B,I)\big) =< A,B,I > \ \forall\ (A,B,I) \in \mathcal{ISOT}(G,\cdot),$$

then

$$\Upsilon\Big(\mathcal{PISOT}(G,\cdot)\Big) = S_n \times S_n \times \{I\}. \ \therefore\ \mathcal{ISOT}(G,\cdot) \succsim S_n \times S_n \times S_n.$$

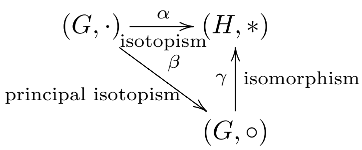
## §4. Smarandache $f,g$-Isotopes of Smarandache loops

**Theorem 4.1.** Let $(G,\cdot)$ and $(H,*)$ be S-groupoids. If $(G,\cdot)$ and $(H,*)$ are S-isotopic, then $(H,*)$ is S-isomorphic to some Smarandache principal isotope $(G,\circ)$ of $(G,\cdot)$.

**Proof.** Since $(G,\cdot)$ and $(H,*)$ are S-isotopic S-groupoids with S-subsemigroups $(G_1,\cdot)$ and $(H_1,*)$, then there exist bijections $U,V,W : (G,\cdot) \rightarrow (H,*)$ such that the triple $\alpha = (U,V,W) : (G,\cdot) \rightarrow (H,*)$ is an isotopism and $\big(G_1\big)A = H_1\ \forall\ A \in \{U,V,W\}$. To prove the claim of this theorem, it suffices to produce a closed binary operation '$*$' on $G$, bijections $X,Y : G \rightarrow G$, and bijection $Z : G \rightarrow H$ so that

- the triple $\beta = (X,Y,I) : (G,\cdot) \rightarrow (G,\circ)$ is a Smarandache principal isotopism and
- $Z : (G,\circ) \rightarrow (H,*)$ is an S-isomorphism or the triple $\gamma = (Z,Z,Z) : (G,\circ) \rightarrow (H,*)$ is an S-isotopism.

Thus, we need $(G, \circ)$ so that the commutative diagram below is true:




because following the proof of transitivity in Theorem 3.1, $\alpha = \beta\gamma$ which implies $(U, V, W) = (XZ, YZ, Z)$ and so we can make the choices; $Z = W$, $Y = VW^{-1}$, and $X = UW^{-1}$ and consequently,

$$x \cdot y = xUW^{-1} \circ VW^{-1} \iff x \circ y = xWU^{-1} \cdot yWV^{-1} \ \forall \ x, y \in G.$$

Hence, $(G, \circ)$ is a groupoid principal isotope of $(G, \cdot)$ and $(H, *)$ is an isomorph of $(G, \circ)$. It remains to show that these two relationships are Smarandache.

Note that $\big((H_1)Z^{-1}, \circ\big) = (G_1, \circ)$ is a non-trivial subsemigroup in $(G, \circ)$. Thus, $(G, \circ)$ is an S-groupoid. So $(G, \circ) \succsim (H, *)$. $(G, \cdot)$ and $(G, \circ)$ are Smarandache principal isotopes because $(G_1)UW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$ and $(G_1)VW^{-1} = (H_1)W^{-1} = (H_1)Z^{-1} = G_1$.

**Corollary 4.1.** Let $(G, \cdot)$ be an S-groupoid with an arbitrary groupoid isotope $(H, *)$. Any such groupoid $(H, *)$ is an S-groupoid if and only if all the principal isotopes of $(G, \cdot)$ are S-groupoids.

**Proof.** By classical result in principal isotopy [[11], III.1.4 Theorem], if $(G, \cdot)$ and $(H, *)$ are isotopic groupoids, then $(H, *)$ is isomorphic to some principal isotope $(G, \circ)$ of $(G, \cdot)$. Assuming $(H, *)$ is an S-groupoid then since $(H, *) \cong (G, \circ)$, $(G, \circ)$ is an S-groupoid. Conversely, let us assume all the principal isotopes of $(G, \cdot)$ are S-groupoids. Since $(H, *) \cong (G, \circ)$, then $(H, *)$ is an S-groupoid.

**Theorem 4.2.** Let $(G, \cdot)$ be an S-quasigroup. If $(H, *)$ is an S-loop which is S-isotopic to $(G, \cdot)$, then there exist S-elements $f$ and $g$ so that $(H, *)$ is S-isomorphic to a Smarandache $f, g$ principal isotope $(G, \circ)$ of $(G, \cdot)$.

**Proof.** An S-quasigroup and an S-loop are S-groupoids. So by Theorem 4.1, $(H, *)$ is S-isomorphic to a Smarandache principal isotope $(G, \circ)$ of $(G, \cdot)$. Let $\alpha = (U, V, I)$ be the Smarandache principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Since $(H, *)$ is a S-loop and $(G, \circ) \succsim (H, *)$ implies that $(G, \circ) \cong (H, *)$, then $(G, \circ)$ is necessarily an S-loop and consequently, $(G, \circ)$ has a two-sided identity element say $e$ and an S-subgroup $(G_2, \circ)$. Let $\alpha = (U, V, I)$ be the Smarandache principal isotopism of $(G, \cdot)$ onto $(G, \circ)$. Then,

$$xU \circ yV = x \cdot y \ \forall \ x, y \in G \iff x \circ y = xU^{-1} \cdot yV^{-1} \ \forall \ x, y \in G.$$

So,

$$y = e \circ y = eU^{-1} \cdot yV^{-1} = yV^{-1}L_{eU^{-1}} \ \forall \, y \in G \text{ and } x = x \circ e = xU^{-1} \cdot eV^{-1} = xU^{-1}R_{eV^{-1}} \ \forall \, x \in G.$$

Assign $f = eU^{-1}, g = eV^{-1} \in G_2$. This assignments are well defined and hence $V = L_f$ and $U = R_g$. So that $\alpha = (R_g, L_f, I)$ is a Smarandache $f, g$ principal isotopism of $(G, \circ)$ onto $(G, \cdot)$. This completes the proof.

**Corollary 4.2.** Let $(G,\cdot)$ be an S-quasigroup(S-loop) with an arbitrary groupoid isotope $(H,*)$. Any such groupoid $(H,*)$ is an S-quasigroup(S-loop) if and only if all the principal isotopes of $(G,\cdot)$ are S-quasigroups(S-loops).

**Proof.** This follows immediately from Corollary 4.1, since an S-quasigroup and an S-loop are S-groupoids.

**Corollary 4.3.** If $(G,\cdot)$ and $(H,*)$ are S-loops which are S-isotopic, then there exist S-elements $f$ and $g$ so that $(H,*)$ is S-isomorphic to a Smarandache $f,g$ principal isotope $(G,\circ)$ of $(G,\cdot)$.

**Proof.** An S-loop is an S-quasigroup. So the claim follows from Theorem 4.2.

## §5. G-Smarandache loops

**Lemma 5.1.** Let $(G,\cdot)$ and $(H,*)$ be S-isotopic S-loops. If $(G,\cdot)$ is a group, then $(G,\cdot)$ and $(H,*)$ are S-isomorphic groups.

**Proof.** By Corollary 4.3, there exist S-elements $f$ and $g$ in $(G,\cdot)$ so that $(H,*)\succsim(G,\circ)$ such that $(G,\circ)$ is a Smarandache $f,g$ principal isotope of $(G,\cdot)$.

Let us set the mapping $\psi := R_{f\cdot g} = R_{fg} \ : \ G\to G$. This mapping is bijective. Now, let us consider when $\psi := R_{fg} \ : \ (G,\cdot)\to(G,\circ)$. Since $(G,\cdot)$ is associative and $x\circ y = xR_g^{-1}\cdot yL_f^{-1}\ \forall\ x,y\in G$, the following arguments are true.

$x\psi\circ y\psi = x\psi R_g^{-1}\cdot y\psi L_f^{-1} = xR_{fg}R_g^{-1}\cdot yR_{fg}L_f^{-1} = x\cdot fg\cdot g^{-1}\cdot f^{-1}\cdot y\cdot fg = x\cdot y\cdot fg = (x\cdot y)R_{fg} = (x\cdot y)\psi\ \forall\ x,y\in G$. So, $(G,\cdot)\cong(G,\circ)$. Thus, $(G,\circ)$ is a group. If $(G_1,\cdot)$ and $(G_1,\circ)$ are the S-subgroups in $(G,\cdot)$ and $(G,\circ)$, then $\big((G_1,\cdot)\big)R_{fg} = (G_1,\circ)$. Hence, $(G,\cdot)\succsim(G,\circ)$.

$\therefore\ (G,\cdot)\succsim(H,*)$ and $(H,*)$ is a group.

**Corollary 5.1.** Every group which is an S-loop is a GS-loop.

**Proof.** This follows immediately from Lemma 5.1 and the fact that a group is a G-loop.

**Corollary 5.2.** An S-loop is S-isomorphic to all its S-loop S-isotopes if and only if it is S-isomorphic to all its Smarandache $f,g$ principal isotopes.

**Proof.** Let $(G,\cdot)$ be an S-loop with arbitrary S-isotope $(H,*)$. Let us assume that $(G,\cdot)\succsim(H,*)$. From Corollary 4.3, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache $f,g$ principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(H,*)\succsim(G,\circ)$. So, $(G,\cdot)\succsim(G,\circ)$.

Conversely, let $(G,\cdot)\succsim(G,\circ)$, using the fact in Corollary 4.3 again, for any arbitrary S-isotope $(H,*)$ of $(G,\cdot)$, there exists a Smarandache $f,g$ principal isotope $(G,\circ)$ of $(G,\cdot)$ such that $(G,\circ)\succsim(H,*)$. Therefore, $(G,\cdot)\succsim(H,*)$.

**Corollary 5.3.** A S-loop is a GS-loop if and only if it is S-isomorphic to all its Smarandache $f,g$ principal isotopes.

**Proof.** This follows by the definition of a GS-loop and Corollary 5.2.

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