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Smarandache Curves in Minkowski Space-time

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Abstract: A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a *Smarandache Curve*. In this paper, we define a special case of such curves and call it *Smarandache* TB_2 *Curves* in the space E_1^4. Moreover, we compute formulas of its Frenet apparatus according to base curve via the method expressed in [3]. By this way, we obtain an another orthonormal frame of E_1^4.

Key Words: Minkowski space-time, Smarandache curves, Frenet apparatus of the curves.

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§1. Introduction

In the case of a differentiable curve, at each point a tetrad of mutually orthogonal unit vectors (called tangent, normal, first binormal and second binormal) was defined and constructed, and the rates of change of these vectors along the curve define the curvatures of the curve in Minkowski space-time [1]. It is well-known that the set whose elements are frame vectors and curvatures of a curve, is called *Frenet Apparatus* of the curves.

The corresponding Frenet's equations for an arbitrary curve in the Minkowski space-time E_1^4 are given in [2]. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a *Smarandache Curve*. We deal with a special Smarandache curves which is defined by the tangent and second binormal vector fields. We call such curves as *Smarandache* TB_2 *Curves*. Additionally, we compute formulas of this kind curves by the method expressed in [3]. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

§2. Preliminary notes

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented. A more complete elementary treatment can be found in the reference [1].

Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standard flat metric given by

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$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null (light-like) if $g(v,v)$=0 and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\alpha(s)$ be a curve in the space-time E_1^4, parametrized by arclength function s. Then for the unit speed space-like curve α with non-null frame vectors the following Frenet equations are given in [2]:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{1}$$

where T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(N,N) = g(B_1,B_1) = 1, g(B_2,B_2) = -1.$$

Here κ, τ and σ are, respectively, first, second and third curvature of the space-like curve α. In the same space, in [3] authors defined a vector product and gave a method to establish the Frenet frame for an arbitrary curve by following definition and theorem.

Definition 2.1 *Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in E_1^4. The vector product in Minkowski space-time E_1^4 is defined by the determinant*

$$a \wedge b \wedge c = - \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{2}$$

where e_1, e_2, e_3 and e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$$e_1 \wedge e_2 \wedge e_3 = e_4 \ , \ e_2 \wedge e_3 \wedge e_4 = e_1 \ , \ e_3 \wedge e_4 \wedge e_1 = e_2 \ , \ e_4 \wedge e_1 \wedge e_2 = -e_3.$$

Theorem 2.2 *Let $\alpha = \alpha(t)$ be an arbitrary space-like curve in Minkowski space-time E_1^4 with above Frenet equations. The Frenet apparatus of α can be written as follows;*

$$T = \frac{\alpha'}{\|\alpha'\|}, \tag{3}$$

$$N = \frac{\|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha'}{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|}, \tag{4}$$

$$B_1 = \mu N \wedge T \wedge B_2, \tag{5}$$

$$B_2 = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{6}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|}{\|\alpha'\|^4} \tag{7}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| . \|\alpha'\|}{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|} \tag{8}$$

and

$$\sigma = \frac{g(\alpha^{(IV)}, B_2)}{\|T \wedge N \wedge \alpha'''\| . \|\alpha'\|}, \tag{9}$$

where μ is taken -1 or $+1$ to make $+1$ the determinant of $[T, N, B_1, B_2]$ matrix.

§3. Smarandache Curves in Minkowski Space-time

Definition 3.1 *A regular curve in E_1^4, whose position vector is obtained by Frenet frame vectors on another regular curve, is called a Smarandache Curve.*

Remark 3.2 Formulas of all Smarandache curves' Frenet apparatus can be determined by the expressed method.

Now, let us define a special form of Definition 3.1.

Definition 3.3 *Let $\xi = \xi(s)$ be an unit space-like curve with constant and nonzero curvatures κ, τ and σ; and $\{T, N, B_1, B_2\}$ be moving frame on it. Smarandache TB_2 curves are defined with*

$$X = X(s_X) = \frac{1}{\sqrt{\kappa^2(s) + \sigma^2(s)}}(T(s) + B_2(s)). \tag{10}$$

Theorem 3.4 *Let $\xi = \xi(s)$ be an unit speed space-like curve with constant and nonzero curvatures κ, τ and σ and $X = X(s_X)$ be a Smarandache TB_2 curve defined by frame vectors of $\xi = \xi(s)$. Then* *(i) The curve $X = X(s_X)$ is a space-like curve.*

(ii) Frenet apparatus of $\{T_X, N_X, B_{1X}, B_{2X}, \kappa_X, \tau_X, \sigma_X\}$ Smarandache TB_2 curve $X = X(s_X)$ can be formed by Frenet apparatus $\{T, N, B_1, B_2, \kappa, \tau, \sigma\}$ of $\xi = \xi(s)$.

Proof Let $X = X(s_X)$ be a Smarandache TB_2 curve defined with above statement. Differentiating both sides of (10), we easily have

$$\frac{dX}{ds_X}\frac{ds_X}{ds} = \frac{1}{\sqrt{\kappa^2(s)+\sigma^2(s)}}(\kappa N + \sigma B_1). \tag{11}$$

The inner product $g(X', X')$ follows that

$$g(X', X') = 1, \tag{12}$$

where $'$ denotes derivative according to s. (12) implies that $X = X(s_X)$ is a space-like curve. Thus, the tangent vector is obtained as

$$T_X = \frac{1}{\sqrt{\kappa^2(s)+\sigma^2(s)}}(\kappa N + \sigma B_1). \tag{13}$$

Then considering Theorem 2.1, we calculate following derivatives according to s:

$$X'' = \frac{1}{\sqrt{\kappa^2+\sigma^2}}(-\kappa^2 T - \tau\sigma N + \kappa\tau B_1 + \sigma^2 B_2). \tag{14}$$

$$X''' = \frac{1}{\sqrt{\kappa^2+\sigma^2}}[\kappa\tau\sigma T + (-\kappa^3 - \kappa\tau^2)N + (\sigma^3 - \tau^2\sigma)B_1 + \kappa\tau\sigma B_2]. \tag{15}$$

$$X^{(IV)} = \frac{1}{\sqrt{\kappa^2+\sigma^2}}[(...)T + (...)N + (...)B_1 + (\sigma^4 - \tau^2\sigma^2)B_2]. \tag{16}$$

Then, we form

$$\|X'\|^2 .X'' - g(X', X'').X' = \frac{1}{\sqrt{\kappa^2+\sigma^2}}[-\kappa^2 T - \tau\sigma N + \kappa\tau B_1 + \sigma B_2]. \tag{17}$$

Equation (17) yields the principal normal of X as

$$N_X = \frac{-\kappa^2 T - \tau\sigma N + \kappa\tau B_1 + \sigma B_2}{\sqrt{-\kappa^4 + \tau^2\sigma^2 + \kappa^2\tau^2 + \sigma^2}}. \tag{18}$$

Thereafter, by means of (17) and its norm, we write first curvature

$$\kappa_X = \sqrt{\frac{-\kappa^4 + \tau^2\sigma^2 + \kappa^2\tau^2 + \sigma^2}{\kappa^2+\sigma^2}}. \tag{19}$$

The vector product $T_X \wedge N_X \wedge X'''$ follows that

$$T_X \wedge N_X \wedge X''' = \frac{1}{A}\left\{\begin{array}{c} [\kappa\sigma(\kappa^2+\sigma^2)(\tau^2-\sigma)T + \kappa\tau\sigma^2(\kappa^2+\sigma)N \\ -\kappa^2\tau\sigma(\kappa^2+\sigma)B_1 + \kappa\tau(\kappa^2+\sigma^2)(\kappa^2+\tau^2)B_2] \end{array}\right\}, \tag{20}$$

where, $A = \frac{1}{\sqrt{(-\kappa^4+\tau^2\sigma^2+\kappa^2\tau^2+\sigma^2)(\kappa^2+\sigma^2)}}$. Shortly, let us denote $T_X \wedge N_X \wedge X'''$ with $l_1 T + l_2 N + l_3 B_1 + l_4 B_2$. And therefore, we have the second binormal vector of $X = X(s_X)$ as

$$B_{2X} = \mu\frac{l_1 T + l_2 N + l_3 B_1 + l_4 B_2}{\sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}}. \tag{21}$$

Thus, we easily have the second and third curvatures as follows:

$$\tau_X = \sqrt{\frac{(-l_1^2 + l_2^2 + l_3^2 + l_4^2)(\kappa^2 + \sigma^2)}{-\kappa^4 + \tau^2\sigma^2 + \kappa^2\tau^2 + \sigma^2}}, \tag{22}$$

$$\sigma_X = \frac{\sigma^2(\sigma^2 - \tau^2)}{(\kappa^2 + \sigma^2)\sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}}. \tag{23}$$

Finally, the vector product $N_X \wedge T_X \wedge B_{2X}$ gives us the first binormal vector

$$B_{1X} = \mu\frac{1}{L}\left\{\begin{array}{c} [(\kappa\sigma l_3 - \sigma^2 l_2 - \tau(\kappa^2 + \sigma^2)l_4]T - \sigma(\kappa^2 l_4 + \sigma l_1)N \\ +\kappa(\kappa^2 l_4 + \sigma l_1)B_1 + [\kappa^2(\sigma l_2 - \kappa^2 l_3) + \tau l_1(\kappa^2 + \sigma^2)]B_2 \end{array}\right\}, \tag{24}$$

where $L = \frac{1}{\sqrt{(-l_1^2+l_2^2+l_3^2+l_4^2)(\kappa^2+\sigma^2)(-\kappa^4+\tau^2\sigma^2+\kappa^2\tau^2+\sigma^2)}}$. □

Thus, we compute Frenet apparatus of Smarandache TB_2 curves.

Corollary 3.1 *Suffice it to say that* $\{T_X, N_X, B_{1X}, B_{2X}\}$ *is an orthonormal frame of* E_1^4.

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