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Smarandachely k-Constrained Number of Paths and Cycles

P. Devadas Rao[1], B. Sooryanarayana[2], M. Jayalakshmi[3]

[1]Department of Mathematics, Srinivas Institute of Technology. Valachil, Mangalore.
Karnataka State, INDIA, Pin 574 143

[2]Department of Mathematical and Computational Studies, Dr.Ambedkar Institute
of Technology, Bangalore, Karnataka State, INDIA, Pin 560 056

[3]Department of Mathematics, Dayanandasagara College Arts Science and Commerce,
Bangalore, Karnataka State, INDIA, Pin 560 078,

Email: devadasrao@yahoo.co.in, dr_bsnrao@yahoo.co.in, jayachatra@yahoo.co.in

Abstract: A *Smarandachely k-constrained labeling* of a graph $G(V,E)$ is a bijective mapping $f: V\cup E \to \{1,2,..,|V|+|E|\}$ with the additional conditions that $|f(u)-f(v)|\geq k$ whenever $uv\in E$, $|f(u)-f(uv)|\geq k$ and $|f(uv)-f(vw)|\geq k$ whenever $u\neq w$, for an integer $k\geq 2$. A graph G which admits a such labeling is called a *Smarandachely k-constrained total graph*, abbreviated as $k-CTG$. The minimum number of isolated vertices required for a given graph G to make the resultant graph a $k-CTG$ is called the *k-constrained number* of the graph G and is denoted by $t_k(G)$. In this paper we settle the open problems 3.4 and 3.6 in [4] by showing that $t_k(P_n)=0$, if $k\leq k_0$; $2(k-k_0)$, if $k>k_0$ and $2n\equiv 1$ or 2 (mod 3); $2(k-k_0)-1$ if $k>k_0$; $2n\equiv 0 \pmod 3$ and $t_k(C_n)=0$, if $k\leq k_0$; $2(k-k_0)$, if $k>k_0$ and $2n\equiv 0$ (mod 3); $3(k-k_0)$ if $k>k_0$ and $2n\equiv 1$ or 2 (mod 3), where $k_0=\lfloor\frac{2n-1}{3}\rfloor$.

Key Words: Smarandachely k-constrained labeling, Smarandachely k-constrained total graph, k-constrained number, minimal k-constrained total labeling.

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§1. Introduction

All the graphs considered in this paper are simple, finite and undirected. For standard terminology and notations we refer [1], [3]. There are several types of graph labelings studied by various authors. We refer [2] for the entire survey on graph labeling. In [4], one such labeling called Smarandachely labeling is introduced. Let $G=(V,E)$ be a graph. A bijective mapping $f: V\cup E\to\{1,2,...,|V|+|E|\}$ is called a *Smarandachely* $k-constrained\ labeling$ of G if it satisfies the following conditions for every $u,v,w\in V$ and $k\geq 2$;

1. $|f(u)-f(v)|\geq k$
2. $|f(u)-f(uv)|\geq k$,

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3. $|f(uv) - f(vw)| \geq k$

whenever $uv, vw \in E$ and $u \neq w$.

A graph G which admits a such labeling is called a *Smarandachely k-constrained total graph*, abbreviated as $k - CTG$. The minimum number of isolated vertices to be included for a graph G to make the resultant graph is a $k - CTG$ is called *k-constrained number of the graph* G and is denoted by $t_k(G)$, the corresponding labeling is called a *minimal k-constrained total labeling* of G.

We recall the following open problems from [4], for immediate reference.

Problem 1.1 *For any integers $n, k \geq 3$, determine the value of $t_k(P_n)$.*

Problem 1.2 *For any integers $n, k \geq 3$, determine the value of $t_k(C_n)$.*

§2. *k*-Constrained Number of a Path

Let $V(P_n) = \{v_1, v_2, . . . , v_n\}$ and $E(P_n) = \{v_iv_{i+1} \mid 1 \leq i \leq n-1\}$. Designate the vertex v_i of P_n as $2i - 1$ and the edge v_jv_{j+1} as $2j$, for each $i, 1 \leq i \leq n$ and $1 \leq j \leq n-1$.

Lemma 2.1 *Let $k_0 = \lfloor \frac{2n-1}{3} \rfloor$ and $S_l = \{3l-2, 3l-1, 3l\}$ for $1 \leq l \leq k_0$. Let f be a minimal k-constrained total labeling of P_n. Then for each $i, 1 \leq i \leq k_0$, there exist a $l, 1 \leq l \leq k_0$ and a $x \in S_l$ such that $f(x) = i$.*

Proof For $1 \leq l \leq k_0$, let S_l=$\{l_1, l_2, l_3\}$, where $l_1 = 3l - 2, l_2 = 3l - 1, l_3 = 3l$. Let $S = \{1, 2, 3, ..., k_0\}$ and f be a minimal k-constrained total labeling of P_n, $2n \equiv 0 \pmod 3$ and $k > k_0$, then by the definition of f it follows that $|f(S_i) \cap S| \leq 1$, for each $i, 1 \leq i \leq k_0 + 1$, otherwise if $f(l_i), f(l_j) \in S$ for $1 \leq i, j \leq 3, i \neq j$, then $|f(l_i) - f(l_j)| < k_0 < k$, a contradiction. Further, if $f(l_j) \neq i$ for any l, j with $1 \leq l \leq k_0, 1 \leq j \leq 3$ for some $i \in S$, then i should be assigned to an isolated vertex. So, span of f will increase, hence f can not be minimal. □

Lemma 2.2 *Let $S_l = \{3l-2, 3l-1, 3l\}$ and f be a minimal k-constrained total labeling of P_n. Let $f(x) = s_1$ and $f(y) = s_2$ for some $x \in S_l$ and $y \in S_{l+1}$ for some $l, 1 \leq l < m \leq k_0$ and $1 \leq s_1, s_2 \leq k_0$, where $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then $y = x + 3$.*

Proof Let x_1, x_2, x_3 be the elements of S_l and x_4, x_5, x_6 be that of S_{l+1} (i.e. if x_1 is a vertex of P_n then x_3, x_5 are vertices and x_2 is an edge x_1x_3 ; x_4 is an edge x_3x_5 and x_6 is incident with x_5 or if x_1 is an edge, then x_1 is incident with x_2; x_2, x_4, x_6 are vertices and x_3 is an edge x_2x_4, x_5 is an edge x_4x_6).

Let f be a minimal k-constrained total labeling of P_n and $S_1, S_2, ..., S_{k_0}$ be the sets as defined in the Lemma 2.1. Let S_α be the set of first k_0 consecutive positive integers required for labeling of exactly one element of S_l for each $l, 1 \leq l \leq k_0$ as in Lemma 2.1. Then each set $S_l, 1 \leq l \leq k_0$ contains exactly two unassigned elements. Again by Lemma 2.1 exactly one of these unassigned element can be assigned by the set S_β containing next possible k_0 consecutive positive integers not in S_α. After labeling the elements of the set $S_l, 1 \leq l \leq k_0$ by the labels in

$S_\alpha \cup S_\beta$, each S_l contains exactly one element unassigned. Thus these elements can be assigned as per Lemma 2.1 again by the set S_γ having next possible k_0 consecutive positive integers not in $S_\alpha \cup S_\beta$.

Let us now consider two consecutive sets S_l, S_{l+1} (Two sets S_i and S_j are said to be consecutive if they are disjoint and there exists $x \in S_i$ and $y \in S_j$ such that xy is an edge). Let $\alpha_1, \alpha_2 \in S_\alpha, x_i \in S_l$ and $x_j \in S_{l+1}$ such that $f(x_i) = \alpha_1$ and $f(x_j) = \alpha_2$ (such α_1, α_2, x_i and x_j exist by Lemma 2.1). Then, as f is a minimal k-constrained total labeling of P_n, it follows that $|j - i| > 2$ implies $j \geq i + 3$. Now we claim that $j = i + 3$. We note that if $i = 3$, then the claim is obvious. If $i \neq 3$, then we have the following cases.

Case 1 $i = 1$

If $j \neq 4$ then

Subcase 1 $j = 5$

By Lemma 2.1, there exists $\beta_1, \beta_2 \in S_\beta$ and $x_r \in S_l, x_s \in S_{l+1}$ such that $f(x_r) = \beta_1$ and $f(x_s) = \beta_2$. Now $f(x_1) = \alpha_1$, $f(x_5) = \alpha_2$ implies $r = 2$ or $r = 3$ (i.e. $f(x_2) = \beta_1$ or $f(x_3) = \beta_1$).

Subsubcase 1 $r = 2$ (i.e. $f(x_2) = \beta_1$)

In this case, $f(x_6) = \beta_2$ (since $f(x_i) = \beta_1$ and $f(x_j) = \beta_2$ implies $|j - i| > 2$) and hence by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$ which is inadmissible as x_3 and x_4 are incident to each other and $|\gamma_1 - \gamma_2| < k_0 < k$.

Subsubcase 2 $r = 3$ (i.e. $f(x_3) = \beta_1$)

Again in this case, $f(x_6) = \beta_2$. So $f(x_2) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$ which is contradiction as x_2 and x_4 are adjacent to each other and $|\gamma_1 - \gamma_2| < k_0 < k$.

Subcase 2 $j = 6$

Now $f(x_1) = \alpha_1, f(x_6) = \alpha_2$ implies $f(x_2) = \beta_1$ or $f(x_3) = \beta_1$.

Subsubcase 1 $f(x_2) = \beta_1$

In this case, $f(x_5) = \beta_2$ and hence by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$ for some $\gamma_1, \gamma_2 \in S_\gamma$, which is a contradiction as x_3 and x_4 are incident to each other.

Subsubcase 2 $f(x_3) = \beta_1$

In this case, $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$ none of them is possible.

Thus we conclude in Case 1 that if $i = 1$, then $j = 4$, so $j = i + 3$.

Case 2 $i = 2$

In this case we have $j \geq i + 3$, so $j \geq 5$. If $j \neq 5$ then $j = 6$. Now $f(x_2) = \alpha_1, f(x_6) = \alpha_2$ implies $f(x_1) = \beta_1$ or or $f(x_3) = \beta_1$.

`Subcase 1:` $f(x_1) = \beta_1$

But then $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$.

Subsubcase 1 $f(x_4) = \beta_2$

In this case, $f(x_4) = \beta_2$ and by Lemma 2.1 $f(x_3) = \gamma_1, f(x_5) = \gamma_2$, which is a contradiction as x_3 and x_5 are adjacent to each other.

Subsubcase 2 $f(x_5) = \beta_2$

In this case, $f(x_5) = \beta_2$ and by Lemma 2.1 $f(x_3) = \gamma_1$ and $f(x_4) = \gamma_2$, which is not possible as x_3 and x_4 are incident to each other.

Subcase 2 $f(x_3) = \beta_1$

In this case, $f(x_4) = \beta_2$ or $f(x_5) = \beta_2$ none of them is possible.

Thus in this case 2, we conclude that if $i = 2$, then $j = 5$, so $j = i + 3$.

Thus, we conclude that the labels in S_α preserves the position in S_l. The similar argument can be extended for the sets S_β and S_γ also. □

Remark 2.3 *Let* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$ *and* l *be an integer such that* $1 \leq l \leq k_0$. *Let* f *be a minimal* k*-constrained total labeling of a path* P_n *and* $S_\alpha = \{\alpha, \alpha+1, \alpha+2, ..., \alpha + k_0 - 1\}$. *Let* $S_l = \{3l-2, 3l-1, 3l\}$ *and* $f(x) = \alpha + i$ *for some* $x \in S_l$ *Then* $f(y) = \alpha + i + k$ *implies* $y \in S_l$.

Proof After assigning the integers 1 to k_0 one each for exactly one element of S_l, for each $l, 1 \leq l \leq k_0$, an unassigned element in the set containing the element labeled by 1 can be labeled by $k+1$. But no unassigned element of any other set can be labeled by $k+1$. Thus, if the label $k+1$ is not assigned to an element of the set whose one of the element is labeled by 1, then it should be excluded for the labeling of the elements of P_n and hence the number of isolated vertices required to make P_n a k-constrained graph will increase. Therefore, every minimal k-constrained total labeling should include label $k+1$ for an element of the set whose one of the element is labeled by 1. After including $k+1$, by continuing the same argument for $k+2, k+3, \cdots, k+k_0$ one by one we can conclude that the label $k+i$ (and then $2k+i$) can be labeled only for the element of the set whose one of the element is labeled by i. □

Remark 2.4 *If* $1 \in f(S_1)$, *then from the above Lemmas* 2.1, 2.2 *and Remark* 2.3, *it is clear that* $l, l+k, l+2k \in f(S_l)$ *for every* $l, 1 \leq l \leq k_0$, *where* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$.

Lemma 2.5 *Let* $S_i = \{3i-2, 3i-1, 3i\}$ *and* f *be a minimal* k*-constrained total labeling of* P_n *such that* $f(x) = s$ *for some* $x \in S_i$ *for some* $i, 1 \leq i \leq k_0$, *where* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. *Then* $f(y) = s+1$ *implies* $y \in S_{l+1}$ *or* $y \in S_{l-1}$ *and hence by Lemma* 2.2 *we have* $|x - y| = 3$.

Proof Suppose the contrary that $y \in S_j$ for some j where $|j - i| > 1$ and $1 \leq j \leq k_0$. Without loss of generality, we now assume that $j > i+1$ (otherwise relabel the set S_m as $S_{k_0 - m}$ for each $l, 1 \leq m \leq k_0$). Now by repeated application of Lemma 2.1 we get the sequence of consecutive sets $S_i, S_{i+1}, S_{i+2}, ..., S_j$ and the sequence of elements $s = s_0, s_1 = s+1, \ldots, s_{j-i} = s+1$ where $s_t \in S_{i+t}$ for each $t, 0 \leq t \leq j$. As $j > i+1$, this sequence of elements (labels) is neither an increasing nor a decreasing sequence. So, there exists a positive integer l such that $s_{l-1} < s_l$ and $s_{l+1} < s_l$. Also, Remark 2.4 $s_{l+k}, s_{l+2k} \in f(S_{i+l})$, $s_{l+1+k}, s_{l+1+2k} \in f(S_{i+l+1})$ and $s_{l-1+k}, s_{l-1+2k} \in f(S_{i+l-1})$. Let $l_1 = 3(i+l) - 2, l_2 = 3(i+l) - 1, l_3 = 3(i+l)$. We now discuss the following 3! cases.

Case 1 $f(l_1) = s_l, f(l_2) = s_l + k, f(l_3) = s_l + 2k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}, f(l_2-3)=s_{l-1}+k, f(l_3-3)=s_{l-1}+2k$ and $f(l_1+3)=s_{l+1}, f(l_2+3)=s_{l+1}+k, f(l_3+3)=s_{l+1}+2k$. So, $|f(l_1-2)-f(l_1)|\geq k \Rightarrow |s_{l-1}+k-s_l|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 2 $f(l_1)=s_l, f(l_2)=s_l+2k, f(l_3)=s_l+k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}, f(l_2-3)=s_{l-1}+2k, f(l_3-3)=s_{l-1}+k$ and $f(l_1+3)=s_{l+1}, f(l_2+3)=s_{l+1}+2k, f(l_3+3)=s_{l+1}+k$. So, $|f(l_1-1)-f(l_1)|\geq k \Rightarrow |s_{l-1}+k-s_l|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 3 $f(l_1)=s_l+k, f(l_2)=s_l, f(l_3)=s_l+2k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+k, f(l_2-3)=s_{l-1}, f(l_3-3)=s_{l-1}+2k$ and $f(l_1+3)=s_{l+1}+k, f(l_2+3)=s_{l+1}, f(l_3+3)=s_{l+1}+2k$. So, $|f(l_1-1)-f(l_1)|\geq k \Rightarrow |(s_{l-1}+2k)-(s_l+k)|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 4 $f(l_1)=s_l+2k, f(l_2)=s_l, f(l_3)=s_l+k$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+2k, f(l_2-3)=s_{l-1}, f(l_3-3)=s_{l-1}+k$ and $f(l_1+3)=s_{l+1}+2k, f(l_2+3)=s_{l+1}, f(l_3+3)=s_{l+1}+k$. So, $|f(l_1-1)-f(l_2)|\geq k \Rightarrow |(s_{l-1}+k)-s_l)|\geq k \Rightarrow |k-(s_l-s_{l-1})|\geq k \Rightarrow s_l-s_{l-1}\leq 0 \Rightarrow s_l\leq s_{l-1}$, a contradiction.

Case 5 $f(l_1)=s_l+k, f(l_2)=s_l+2k, f(l_3)=s_l$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+k, f(l_2-3)=s_{l-1}+2k, f(l_3-3)=s_{l-1}$ and $f(l_1+3)=s_{l+1}+k, f(l_2+3)=s_{l+1}+2k, f(l_3+3)=s_{l+1}$. So, $|f(l_3+1)-f(l_3)|\geq k \Rightarrow |(s_{l+1}+k)-s_l)|\geq k \Rightarrow |k-(s_l-s_{l+1})|\geq k \Rightarrow s_l-s_{l+1}\leq 0 \Rightarrow s_l\leq s_{l+1}$, a contradiction.

Case 6 $f(l_1)=s_l+2k, f(l_2)=s_l+k, f(l_3)=s_l$.

In this case by Lemma 2.2 it follows that $f(l_1-3)=s_{l-1}+2k, f(l_2-3)=s_{l-1}+k, f(l_3-3)=s_{l-1}$ and $f(l_1+3)=s_{l+1}+2k, f(l_2+3)=s_{l+1}+k, f(l_3+3)=s_{l+1}$. So, $|f(l_3+1)-f(l_2)|\geq k \Rightarrow |(s_{l+1}+2k)-(s_l+k))|\geq k \Rightarrow |k-(s_l-s_{l+1})|\geq k \Rightarrow s_l-s_{l+1}\leq 0 \Rightarrow s_l\leq s_{l+1}$, a contradiction. □

Lemma 2.6 *Let P_n be a path on n vertices and $k_0=\lfloor\frac{2n-1}{3}\rfloor$. Then $t_k(P_n)\geq 2(k-k_0)-1$ whenever $2n\equiv 0(mod3)$ and $k>k_0$.*

Proof For $1\leq l\leq k_0$, let S_l=$\{l_1,l_2,l_3\}$, where $l_1=3l-2, l_2=3l-1, l_3=3l$. Let $S_{k_0+1}=\{2n-2,2n-1\}$ and $T=\{1,2,3,...,k_0\}$. Let f be a minimal k-constrained total labeling of $P_n, 2n\equiv 0$ (mod 3) and $k>k_0$, then by Lemma 2.1, we have $|f(S_i)\cap T|=1$ for each i (i.e. exactly one element of S_i mapped to distinct element of T for each $i, 1\leq i\leq k_0$) and $f(l_j)=m\in T$ for some $j, 1\leq j\leq 3$, then for other element l_i of $S_l, i\neq j$, we have $|f(l_i)-f(l_j)|\geq k$ implies $f(l_i)\geq k+m$. Thus f excludes the elements of the set $T_1=\{k_0+1,k_0+2,...,k\}$ for the next assignments of the elements of $S_l, l\neq k_0+1$.

Let $f(l_i)=t$ for some $t\in T$, where $l_i\in S_l$. Then for the minimum span$f$, by Remark 2.3 $f(l_j)=k+t$ for $i\neq j$ and $l_j\in S_l$.

Again by Lemma 2.3, we get $|f(S_i)\cap T'|=1$, for each $i, 1\leq i\leq k_0$, where $T'=\{k+1,k+$

$2, ..., k+k_0\}$. Further, if f assigns each element of S to exactly one element of $S_l, 1 \le l \le k_0$, for the next assignments, f should leaves all the elements of the set $T_2 = \{k+k_0+1, k+k_0+2, ..., 2k\}$. The above arguments show that while assigning the labels for the elements of P_n not in S_{k_0+1}, f leaves at least $2(k-k_0)$ elements which are in the set $T_1 \cup T_2$.

In view of Lemma 2.2, there are only two possibilities for the assignments of elements of S_{k_0+1} depending upon whether f assigns an element of T_1 to an element of S_{k_0+1} or not.

Let us now consider the first case. Let $x \in S_{k_0+1}$ such that $f(x) = t$ for some $t \in T_1$.

Claim $x = 2n - 1$

If not, $f(2n-2) = t$, but then $f(2n-3) \notin T \cup T_1$ and $f(2n-4) \notin T \cup T_1$. Then by Lemma 2.2 $f(2n-5) \in T \cup T_1$ and by Lemma 2.5 $f(2n-5) = t-1$. Then again as above $f(2n-8) = t-2$. Continuing this argument, we conclude that $f(1) = 1$ and $f(4) = 2$. But then, by above argument, we get $f(x) = k+1$ and $f(x+3) = k+2$ for some $x \in S_1$ and $x \in \{2, 3\}$. So, $|f(x) - f(4)| = |k+1-2| \ngeq k$ and $|4-x| \le 2$, a contradiction. Hence the claim.

By the above claim we get $f(2n-1) \in T_1$. We now suppose that $f(2n-2) \notin T_2$(note that $f(2n-2) \notin T \cup T_1$), then by above argument for the minimality of f we have $f(2n-2) = k+k_0+1$ and hence $f(1) = k+1$ and $f(2) = 1$. So, by Lemma 2.5, $f(4) = k+2$ and $f(5) = 2$. So, $f(3) \neq 2k+1$ (Since $|f(3) - f(4)| = |2k+1-(k+2)| \ngeq k$, which is inadmissible). This shows that f includes either at most one element of $T_1 \cup T_2$ to label the elements of S_{k_0+1} or leaves one more element namely $2k+1$ to label the elements of P_n (Since the label $2k+1$ is possible only for the element in S_1. Thus f leaves at least $2(k-k_0) - 1$ elements.

If the second case follows then the result is immediate because f leaves $(k-k_0)$ elements in the first round of assignment and uses exactly one element of T_2 in the second round. □

Remark 2.7 *In the above Lemma* 2.6 *if* $2n \not\equiv 0(mod\ 3)$ *, then* $t_k(P_n) \ge 2(k-k_0)$.

Proof If the hypothesis hold, then $S_{k_0+1} = \emptyset$ or $S_{k_0+1} = \{2n-1\}$. In the first case, if $S_{k_0+1} = \emptyset$, then by the proof of the Lemma we see that any minimal k-constrained total labeling f should leave exactly $2(k-k_0)$ integers for the labeling of the elements of the path P_n. In the second case when $S_{k_0+1} = 2n-1$, by Lemma 2.5 $f(2n-1) = k_0+1$ (we can assume that $f(1) \in f(S_1)$ because only other possibility by Lemma 2.5 is that the labeling of elements of P_n is in the reverse order, in such a case relabel the sets S_l as S_{k_0-l}). But then, again by Lemma 2.2 and Lemma 2.5 it forces to take $f(1) = 1$ and $f(4) = 2$ hence by Remark 2.4, $f(x) = k+1$ only if $x = 2$ or $x = 3$. In either of the cases $|f(4) - f(x)| \ngeq k$, a contradiction. Hence neither k_0+1 nor $k+1$ can be assigned. Further, if k_0+1 is not assigned, then in the similar way we can argue that either $k+k_0+1$ or $2k+1$ can not be assigned while assigning the second elements of each of the sets $S_l, 1 \le l \le k_0$. Thus, in both the cases f should leave at least $2(k-k_0)$ integers for the assignment of P_n, whenever $2n \not\equiv 0$ (mod 3). □

Theorem 2.8 *Let* P_n *be a path on* n *vertices and* $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. *Then*

$$t_k(P_n) = \begin{cases} 0 \quad if \quad k \le k_0, \\ 2(k-k_0) - 1 \quad if \quad k > k_0 \quad and \quad 2n \equiv 0(mod\ 3), \\ 2(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2(mod\ 3). \end{cases}$$

Proof If $k \le k_0$, then the result follows by Theorem 3.3 of [4]. Consider the case $k > k_0$.

Case i $2n \equiv 0 \pmod 3$

By Lemma 2.6 we have $t_k(P_n) \ge 2(k - k_0) - 1$. Now, the function $f : V(P_n) \cup E(P_n) \cup \overline{K}_{2(k-k_0)-1} \to \{1, 2, \ldots, 2(n + k - k_0) - 2\}$ defined by $f(1) = 2k + 1, f(2) = k + 1, f(3) = 1$ and $f(i) = f(i - 3) + 1$ for all $i, 4 \le i \le 2n - 3, f(2n - 2) = 2k + 1 + k_0$, $f(2n - 1) = k + 1 + k_0$ and the vertices of $\overline{K}_{2(k-k_0)-1}$ to the remaining, is a Smarandachely k-constrained labeling of the graph $P_n \cup \overline{K}_{2(k-k_0)-1}$. Hence $t_k(P_n) \le 2(k - k_0) - 1$.

Case ii $2n \not\equiv 0 \pmod 3$

By Remark 2.7 we have $t_k(P_n) \ge 2(k - k_0)$. On the other hand, the function $f : V(P_n) \cup E(P_n) \cup \overline{K}_{2(k-k_0)} \to \{1, 2, \ldots, 2(n+k-k_0)-1\}$ defined by $f(1) = 2k+1, f(2) = k+1, f(3) = 1$, $f(i) = f(i - 3) + 1$ for all $i, 4 \le i \le 2n - 1$ and the vertices of $\overline{K}_{2(k-k_0)}$ to the remaining, is a Smarandachely k-constrained labeling of the graph $P_n \cup \overline{K}_{2(k-k_0)}$. Hence $t_k(P_n) \le 2(k - k_0)$.□

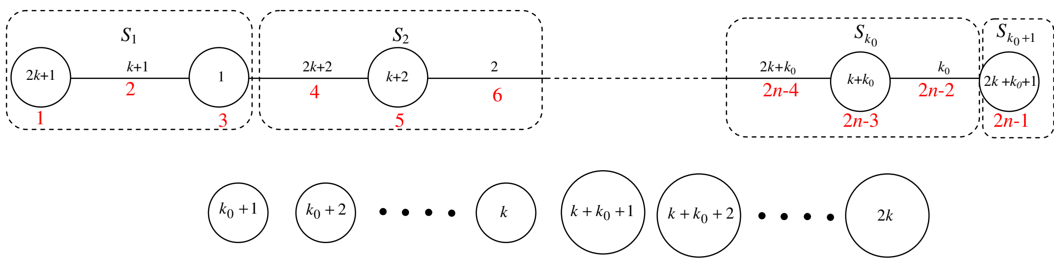

Figure 1: A k-constrained total labeling of the path $P_n \cup \overline{K}_{2(k-k_0)}$, where $2n \equiv 2 \pmod 3$.

§3. k-Constrained Number of a Cycle

Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} \mid 1 \le i \le n - 1\} \cup \{v_n v_1\}$. Due to the symmetry in C_n, without loss of generality, we assume that the integer 1 is labeled to the vertex v_1 of C_n. Define $S_\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$, for all $\alpha \in Z^+, 1 \le \alpha \le k_0$, where $k_0 = \lfloor \frac{2n-1}{3} \rfloor$ and $\alpha_1 = v_{\frac{3\alpha-1}{2}}, \alpha_2 = v_{\frac{3\alpha-1}{2}} v_{\frac{3\alpha+1}{2}}, \alpha_3 = v_{\frac{3\alpha+1}{2}}$ for all odd α and if α is even, then and $\alpha_1 = v_{\frac{3\alpha}{2}-1} v_{\frac{3\alpha}{2}}, \alpha_2 = v_{\frac{3\alpha}{2}}, \alpha_3 = v_{\frac{3\alpha}{2}} v_{\frac{3\alpha}{2}+1}$.

Case 1 $2n \equiv 0 \pmod 3$

In this case set of elements (edges and vertices) of C_n is $S_1 \cup S_2 \cup \cdots \cup S_{k_0} \cup S_{k_0+1}$, where $S_{k_0+1} = \{v_{n-1}v_n, v_n, v_n v_1\}$.

We now assume the contrary that $t_k(C_n) < 2(k - k_0)$. Then there exists a minimal k-constrained labeling f such that span f is less that $k_0 + 2k + 3$ (since span f = number of vertices + edges + $t_k(C_n) < 3(k_0 + 1) + 2(k - k_0)$). Now our proof is based on the following observations.

Observation 3.1 Let L_1 be the set of first possible consecutive integers (labels) that can be assigned for the elements of C_n. Then exactly one element of each set $S_\alpha, 1 \le \alpha \le k_0 + 1$, can receive one distinct label in L_1 and for the minimum span all the labels in L_1 to be assigned. Thus $|L_1| = k_0 + 1$.

Observation 3.2 The labels in L_1 can be assigned only for the elements of S_α in identical places (i.e. $\alpha_i \in S_\alpha$ receives $f(\alpha_i) \in L_1$ and $\beta_j \in S_\beta$ receives $f(\beta_j) \in L_1$ if and only if $i = j$ for all α, β). In fact, since $\alpha_1 = 1$, when $\alpha = 1$, we get $f(\beta_1) \in L_1$, where $\beta = k_0 + 1$, hence $f(\gamma_1) \in L_1$, where $\gamma = k_0$, and so on $\cdots$.

Observation 3.3 The observation 3.2 holds for next labelings for the remaining unlabeled elements also.

Observation 3.4 Since the smallest label in L_1 is 1, by observation 3.1, it follows that the largest label in L_1 is $k_0 + 1$ and next minimum possible integer(label) in the set L_2, consisting of consecutive integers used for the labeling of elements unassigned by the set L_1, is $k + 2$ (we observe that $k + i$, for $k_0 - k + 1 < i < 1$ can not be used for the labeling of any element in the set $S_\alpha, 1 \leq \alpha \leq k_0 + 1$ (since an element of each of S_α has already received a label x in L_1, $1 \leq x \leq k_0 + 1$ and $(k + i) - (x) = k + (i - x) < k$. Also if $k + 1$ is assigned, then $k + 1$ is assigned only to 2^{nd} or 3^{rd} element (viz α_2 or α_3, where $\alpha = 1$) of S_1, but then difference of labels of first element of S_2 labeled by an integer in L_1 (which is greater than 1) with $k + 1$ differs by at most by $k - 1$).

Observation 3.5 By observation 3.4 it follows that the minimum integer label in L_2 is $k + 2$, so the maximum integer label is $k + k_0 + 2$.

Observation 3.6 Let L_3 be the set of next consecutive integers which can be used for the labeling of the elements not assigned by $L_1 \cup L_2$. Then, as span is less than $k_0 + 2k + 3$, the maximum label in L_3 is at most $k_0 + 2k + 2$ and hence the minimum is at most $2k + 2$.

We now suppose that $f(\alpha_i) \in L_3$ and $f(\alpha_i) = \min L_3$, for some $\alpha, 1 \leq \alpha \leq k_0 + 1$. Then, as $f(\alpha_i) = \min L_3$, $f(\alpha_i) = 2k + j$ for some $j \leq 2$. Further, as $f(\alpha_i) \notin L_2$, we have $k_0 + 2 - k \leq j$. Combining these two we get $k_0 + 2 - k \leq j \leq 2$.

Subcase 1 $i = 2$

In this case $f(\alpha_2) \in L_3$ and already $f(\alpha_1) \in L_1$, so $f(\alpha_3) \in L_2$ and hence $f(\beta_3) \in L_2$ (by Observation 3.2), where $\beta = \alpha - 1$ (or $\beta = k_0 + 1$ if $\alpha = 1$). Thus, $f(\beta_3) = k + l$ for some $l, 2 \leq l \leq k + 2 + k_0$

Now $|f(\alpha_2) - f(\beta_3)| = |(2k + j) - (k + l)| = |k + (j - l)| \geq k$ implies that $j - l \geq 0$ hence $j \geq l$. But $j \leq 2 \leq l$ implies $j = l = 2$. Therefore, $f(\alpha_2) = 2k + 2$ and $f(\beta_3) = k + l = k + 2 = \min L_2$

In this case $f(\alpha_3) \in L_2$ implies that $f(\alpha_3) = k + m$, for some $m > 2$. So, $|f(\alpha_2) - f(\alpha_3)| = |(2k + 2) - (k + m)| = |k + (2 - m)| < k$ as $m > 2$, which is a contradiction.

Subcase 2 $i = 3$

In this case $f(\alpha_3) \in L_3$ and already $f(\alpha_1) \in L_1$, so $f(\alpha_2) \in L_2$ and hence $f(\beta_2) \in L_2$ (by Observation 3.2), where $\beta = \alpha - 1$ (or $\beta = 1$ if $\alpha = k_0 + 1$). Thus, $f(\beta_2) = k + l$ for some $l, 2 \leq l \leq k + 2 + k_0$.

Now $|f(\alpha_3) - f(\beta_2)| = |(2k + j) - (k + l)| = |k + (j - l)| \geq k$ implies that $j - l \geq 0$ hence

$j \geq l$. But $j \leq 2 \leq l$ implies $j = l = 2$. Therefore, $f(\alpha_3) = 2k+2$ and $f(\beta_2) = k+l = k+2 =$ min L_2.

In this case $f(\alpha_2) \in L_2$ implies that $f(\alpha_2) = k+m$, for some $m > 2$. So, $|f(\alpha_3) - f(\alpha_2)| = |(2k+2) - (k+m)| = |k + (2-m)| < k$ as $m > 2$, which is a contradiction.

Hence in either of the cases we get $t_k(C_n) \geq 2(k - k_0)$.

Case 2 $2n \not\equiv 0 \pmod 3$

Let f be a minimal k-constrained total labeling of C_n. Let L_1, L_2, L_3 be the sets as defined as in Observations 3.1, 3.4 and 3.6 above. Let L_4 be the set of possible consecutive integers used for labeling the elements of C_n which are not assigned by the set $L_1 \cup L_2 \cup L_3$.

We first take the case $2n \equiv 1 \pmod 3$. If possible we now again assume the contrary that $t_k(C_n) < 3(k - k_0)$. Then it follows that span f is less than $3k+1$.

Observation 3.7 Since minimum label in L_1 is 1 and f is a minimal k-constrained labeling, we have $f(x) \geq k+1$ for all x such that $f(x) \in L_2$.

We have $f(\alpha_1) = 1$ for $\alpha = 1$. Let β be the smallest index such that $f(\beta_1) \in L_1$ and $f(\gamma_1) \notin L_1$, where $\gamma = \beta + 1$ (such index β exists because $f(\alpha_1) = 1$ for $\alpha = 1$ and γ exists because $2n \not\equiv 0 \pmod 3$, the elements labeled by L_1 differ by it position by exactly multiples of 3 apart on either sides of the element labeled by 1). Now consider the set $S = \{\beta_2, \beta_3, \gamma_1\}$. None of the elements of S can be labeled by any the label in L_1 and no two of them receive the label for a single set L_i, for any $i, 2 \leq i \leq 4$. Let s_1, s_2, s_3 be the elements of S arranged accordingly $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$.

Since span $f \leq 3k$, we have $f(s_3) \leq 3k$, so $f(s_2) \leq 2k$ and hence $f(s_1) \leq k$, which is a contradiction (follows by Observation 3.7). Hence for any minimal k-constrained labeling f we get $t_k(C_n) \geq 3(k - k_0)$ whenever $2n \equiv 1 \pmod 3$.

We now take the case $2n \equiv 2 \pmod 3$. If possible we now again assume the contrary that $t_k(C_n) < 3(k - k_0)$. Then it follows that span f is less than or equal to $3k+1$. The element of C_n is the set $S_1 \cup S_2 \cup \cdots \cup S_{k_0} \cup S_{k_0+1}$, where $S_{k_0+1} = \{v_n, v_n v_1\}$. We now claim that the label of the first element namely α_1 of the set S_α is in the set L_1 for all $\alpha, 1 \leq \alpha \leq k_0$ if and only if $k_0 > 2$.

Suppose that α is the least positive index such that $f(\alpha_1) \notin L_1$ and $1 < \alpha \leq k_0$. Then for all β such that $1 \leq \beta < \alpha, f(\beta_1) \in L_1$. Let $\beta = \alpha - 1$. Consider the set $S = \{\beta_2, \beta_3, \alpha_1\}$. Let s_1, s_2, s_3 be the rearrangements of the elements in the set S such that $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$ respectively.

Since $f(s_3) \in L_4$ and span f is less than or equal to $3k+1$ it follows that $f(s_3) \leq 3k+1$ and hence $f(s_2) \leq 2k+1, f(s_1) \leq k+1$. But, the least element in L_1 is 1 implies that the least element in L_2 is greater than or equal to $k+1$, so $f(s_1) \geq k+1$. Therefore, $f(s_1) = k+1$, so that $f(s_2) = 2k+1$ and $f(s_3) = 3k+1$. There are two possible cases depending on $s_3 \in S_\alpha$ or not. Before considering these cases we make the the following observations.

Observation 3.8 Since $f(\alpha_1) \in L_4$, we find $f(\alpha_1) = 3k+1$ for any $\alpha > 1$. Suppose for any δ, $\delta > \alpha$, if $f(\delta_1) \in L_1$, then for any γ, $\gamma > \delta$, we find $f(\gamma_1) \in L_1$. In fact, for $\gamma > \delta$, if $f(\gamma_1) \notin L_1$ and $f(\eta_1) \in L_1$ for $\eta = \gamma - 1$, then sequence s_1, s_2, s_3 of the elements of the set $S = \{\eta_2, \eta_3, \gamma_1\}$

taken accordingly as $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$ as above, we get $f(s_3) \leq 3k$ (since $3k+1$ is already assigned). Therefore, $f(s_2) \leq 2k$ and hence $f(s_1) \leq k$, which is imposable (since $f(s_1 \notin L_1)$.

This shows that if $f(\delta_1) \in L_1$, where $\delta = \alpha + 1$, we arrive at the situation that $f(\eta_1) \in L_1$, where $\eta = k_0$.

Now taking the set $\{\eta_2, \eta_3, v_n\}$ and rearranging these elements as s_1, s_2, s_3 such that $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$, we get $f(s_1) \leq k$ which is again a contradiction.

Observation 3.9 Observation 3.8 shows that $f(\delta_1) \notin L_1$ for any $\delta, \alpha < \delta \leq k_0$.

Observation 3.10 Starting from the vertex v_1, consider the sets $\acute{S_1} = \{v_1, v_1v_n, v_n\}$, $\acute{S_2} = S_{k_0}$, $\acute{S_3} = S_{k_0-1}, \ . \ . \ . \ , \acute{S}_{k_0-\delta+2} = S_\delta$. By taking these sets, we arrive at the conclusion, as in Observation 3.8, that $f(\delta_3) \in L_1$ for every $\delta > \alpha$.

We now continue the main proof for the first case $s_3 \in S_\alpha$. In this case $s_3 = \alpha_1$, therefore $s_1 \in S_\beta$. But $f(s_3) \in L_4$ implies that $f(s_3) \leq 3k+1$, so $f(s_2) \leq 2k+1$ and hence $f(s_1) \leq k+1$. On the other hand $f(\beta_1) \in L_1$ implies that $f(\beta_2)$ or $f(\beta_3)$ is greater than or equal to $k+1$ (since $\min L_1 = 1$), that is, $f(s_1) \geq k+1$. Thus, $f(s_1) = k+1$. This yields $f(\beta_1) = 1$, so $\beta = 1$ and $\alpha = 2$. Also $f(s_2) = 2k+1$ and $f(s_3) = 3k+1$.

Let us now suppose that $\alpha < k_0$. Then there exists an index δ such that $\delta = \alpha + 1 \leq k_0$.

If $f(\beta_2) = 2k+1, f(\beta_3) = k+1$, then $f(\alpha_2) \geq 2k+1$ (since $f(\beta_3) = k+1$) and $f(\alpha_2) \leq 2k+1$ (since $f(\alpha_1) = 3k+1$). So, $f(\alpha_2) = 2k+1$ and hence $f(\alpha_2) = f(\beta_2)$ which is not possible (since $\alpha \neq \beta$).

If $f(\beta_2) = k+1, f(\beta_3) = 2k+1$, then $f(\alpha_2) \leq k+1$ implies $f(\alpha_2) \in L_1$ (since $f(\alpha_2) \neq k+1 = f(\beta_2)$). Further by Observation 3.10, we have $f(\delta_3) \in L_1$. Consider the set $\{\alpha_3, \delta_1, \delta_2\}$ (we note that none of the elements of this set is labeled by the set L_1) and let s_1, s_2, s_3 be the elements of this set taken in order such that $f(s_1) \in L_2, f(s_2) \in L_3, f(s_3) \in L_4$. Since $3k+1$ is already assigned we get $f(s_3) \leq 3k$ and hence as above $f(s_1) \leq k$, which is a contradiction to the fact $f(s_1) \notin L_1$.

We now continue the main proof for the second case $s_3 \notin S_\alpha$. In this case $s_3 \in S_\beta$. Now by assumption we have $f(\alpha_3) \in L_1$ and $k+1$ is already labeled for an element of $S_\beta = S_1$, therefore, $f(\alpha_1) = 2k+1$. Now by Observation 3.10, $f(\delta_3) \in L_1$, where $\delta = \alpha + 1$. If $f(\alpha_2) \in L_1$, then by taking the set $\{\alpha_3, \delta_1, \delta_2\}$ and arranging as above we can show that one of these elements must be labeled by an element of the set L_4 and hence that label should be at most $3k$, so the smallest label of the element of the set is less than or equal k, a contradiction to the fact that the smallest label is not in L_1. Thus, $f(\alpha_2) \notin L_1$.

If $f(\beta_3) = 3k+1$, then $f(\alpha_2) \in L_2$, and hence $f(\alpha_2) \geq k+2$, which is not possible because $f(\alpha_1) = 2k+1$. Therefore, $f(\beta_2) = 3k+1$ and $f(\beta_3) = k+1$. But then, only possibility is that $f(\alpha_2) \in L_4$ implies that $f(\alpha_2) \leq 3k$, which is impossible because $f(\alpha_1) = 2k+1$. Hence the claim.

By the above claim we have either first element of all the sets $S_1, S_2, \ . \ . \ . S_{k_0}$ are labeled by the elements of the set L_1 or the graph is the cycle C_4. For the graph C_4, it is easy to observe that no three consecutive integers can be used for the labeling and hence each of the sets L_1, L_2, L_3 and L_4 should have at most two elements. Thus, span $f \geq 3k+2$. The equality holds by the following Figure 2.

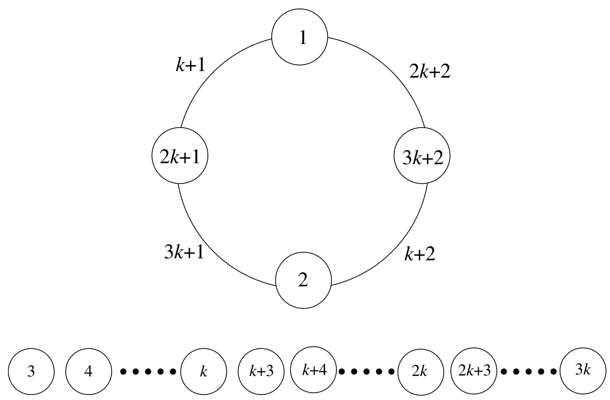

Figure 2: A k-constrained total labeling of the graph $C_4 \cup \overline{K}_{3k-6}$

If the graph is not C_4, then consider the set $T = \{v_{n_1}, v_{n-1}v_n, v_n, v_nv_1\}$. Since $f(v_{n-2}v_{n-1}) \in L_1$ (follows by Observation 3.10) and $f(v_1) = 1 \in L_1$ (follows by the assumption) none of the elements of the set T is labeled by the set L_1 and exactly two elements namely v_{n-1} and v_nv_1 are labeled by same set.

If $f(v_{n-1})$ and $f(v_nv_1)$ are in L_2, then either $f(v_{n-1}v_n)$ and $f(v_n)$ is in L_4. Suppose $f(v_{n-1}v_n)$ (similarly $f(v_n) \in L_4$), then $f(v_n) \in L_3$ ($f(v_{n-1}v_n) \in L_3$), so $f(v_{n-1}v_n) \leq 3k+1$ and hence $f(v_n) \leq 2k+1$. Therefore both $f(v_{n-1})$ and $f(v_nv_1)$ must be less than or equal to $k+1$, which is not possible because minimum of L_2 is $k+1$.

If $f(v_{n-1})$ and $f(v_nv_1)$ are in L_3, then $f(v_n) \in L_4$ (or $f(v_{n-1}v_n) \in L_4$) so $f(v_nv_1) \leq 2k+1$ and $f(v_{n-1}) \leq 2k+1$ (since $f(v_n) \leq 3k+1$). Therefore, at least one of $f(v_nv_1)$ or $f(v_{n-1})$ is less than or equal to $2k$, which yields that $f(v_{n-1}v_n) \leq k$ ($f(v_n) \leq k$). Thus, either $f(v_{n-1}v_n)$ or $f(v_n)$ are in L_1, a contradiction.

If $f(v_{n-1})$ and $f(v_nv_1)$ are in L_4, then at least one of them must be less than $3k+1$. Hence either $f(v_n)$ or $f(v_{n-1}v_n)$ is less than or equal to k (as above), which is again a contradiction.

Thus, we conclude

Lemma 3.11 *Let C_n be a cycle on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then*

$$t_k(C_n) \geq \begin{cases} 0 \quad if \quad k \leq k_0, \\ 2(k-k_0) \quad if \quad k > k_0 \quad and \quad 2n \equiv 0 \ (mod\ 3), \\ 3(k-k_0) \quad if \ k > k_0 \quad and \quad 2n \equiv 1 \ or \ 2 (mod\ 3). \end{cases}$$

Now to prove the reverse inequality, designate the vertex v_i of C_n as $2i-1$ and the edge v_jv_{j+1} as $2j$, v_nv_1 as $2n$. For each $i, 1 \leq i \leq n$ and $1 \leq j \leq n-1$ and for the case $2n \equiv 0$ (mod 3), define a function $f: V(C_n) \cup E(C_n) \cup V(\overline{K}_{2(k-k_0)}) \to \{1, 2, 3, \ldots, 2k+k_0+3\}$ by $f(1) = 1, f(2) = k+2, f(3) = 2k+3$, $f(i) = f(i-3)+1$, for $4 \leq i \leq 2n$ and the vertices of $\overline{K}_{2(k-k_0)}$ to the remaining.

The function f serves as a Smarandachely k-constrained labeling of the graph $C_n \cup \overline{K}_{2(k-k_0)}$. Hence $t_k(C_n) \le 2(k - k_0)$.

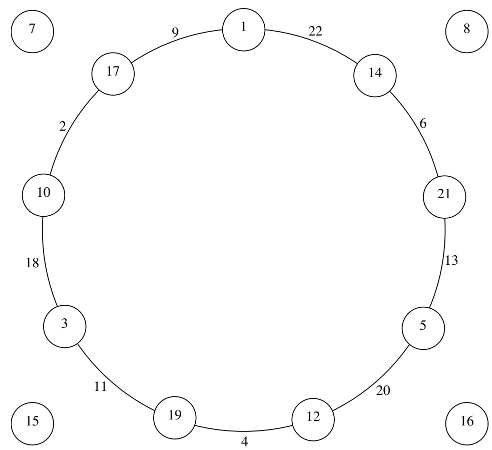

Figure 3: A 7-constrained total labeling of the graph $C_9 \cup \overline{K}_{2(k-k_0)}$

For the case $2n \equiv 1 \pmod 3$, define a function $f : V(C_n) \cup V(C_n) \cup V(\overline{K}_{3(k-k_0)}) \to \{1, 2, 3, . . . , 3k + 1\}$ by $f(1) = 1, f(2) = 2k + 2, f(3) = k + 2$, $f(i) = f(i - 3) + 1$ for $4 \le i \le 2n - 4$, $f(2n - 3) = k_0$, $f(2n - 2) = 3k + 1$, $f(2n - 1) = 2k + 1$, $f(2n) = k + 1$ and the vertices of $\overline{K}_{3(k-k_0)}$ to the remaining.

The function f serves as a Smarandachely k-constrained labeling of the graph $C_n \cup \overline{K}_{3(k-k_0)}$. Hence $t_k(C_n) \le 3(k - k_0)$.

For the case $2n \equiv 2 \pmod 3$, define a function $f : V(C_n) \cup V(C_n) \cup V(\overline{K}_{3(k-k_0)}) \to \{1, 2, 3, . . . , 3k + 2\}$ by $f(1) = 1, f(2) = k + 2, f(3) = 2k + 3$, $f(i) = f(i - 3) + 1$, for $4 \le i \le 2n - 6$, $f(2n - 5) = 3k + 1$, $f(2n - 4) = k_0$, $f(2n - 3) = 2k + 1$, $f(2n - 2) = 3k + 2$, $f(2n - 1) = k + 1$, $f(2n) = 2k + 2$ the vertices of $\overline{K}_{3(k-k_0)}$ to the remaining.

The function f serves as a Smarandachely k-constrained labeling of the graph $C_n \cup \overline{K}_{3(k-k_0)}$. Hence $t_k(C_n) \le 3(k - k_0)$.

Hence, in view of Lemma 3.11, we get

Theorem 3.12 *Let C_n be a cycle on n vertices and $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. Then*

$$t_k(C_n) = \begin{cases} 0 \ \ if \ \ k \le k_0, \\ 2(k - k_0) \ \ if \ \ k > k_0 \ \ and \ \ 2n \equiv 0 \ (mod\ 3), \\ 3(k - k_0) \ \ if \ k > k_0 \ \ and \ \ 2n \equiv 1 \ or \ 2 (mod\ 3). \end{cases}$$

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