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On the value distribution of the Smarandache multiplicative function[1]

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Abstract The main purpose of this paper is using the elementary method to study the value distribution property of the Smarandache multiplicative function, and give an interesting asymptotic formula for it.

Keywords Smarandache multiplicative function, value distribution, asymptotic formula.

§1. Introduction

Let n and m are two positive integers with $(n,\ m) = 1$, the famous Smarandache multiplicative function $f(n)$ is defined as following:

$$f(nm) = \max\{f(n),\ f(m)\}.$$

It is easy to know that Smarandache multiplicative function is not a multiplicative function, in fact, for two different primes p and q,

$$f(p^\alpha q^\beta) \neq f(p^\alpha)f(q^\beta).$$

About this function and many other Smarandache type functions, many scholars had studied them properties, see [1], [2], [3] and [4]. For example, professor Henry Bottomley [5] had considered eleven particular families of interrelated multiplicative functions, many of which are listed in the Smarandache's problem. Tabirca [6] proved an interesting properties about the Smarandache multiplicative function: If $f(n)$ be the Smarandache multiplicative function, then

$$g(n) = \min\{f(d):\ d|n,\ d \in N\}$$

is the Smarandache multiplicative function too.

For any fixed positive integer n, let $p(n)$ denotes the greatest prime divisor of n, $S(n) = \min\{m:\ m \in N,\ n|m!\}$ be the Smarandache function. From this, we know that $p(n)$ and $S(n)$ are the Smarandache multiplicative functions. Dr.Z.F.Xu [7] deduced that for any real number $x > 0$,

$$\sum_{n\leq x}(S(n)-p(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

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where $\zeta(s)$ is the Riemann zeta-function.

Now, for any prime p and positive integer α, we define $f(p^\alpha) = p^{\frac{1}{\alpha}}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime power factorizations of n, then from the definition of $f(p^\alpha)$ we have

$$f(n) = \max_{1\le i\le r}\{f(p_i^{\alpha_i})\} = \max_{1\le i\le r}\left\{p_i^{\frac{1}{\alpha_i}}\right\}.$$

It is clear that $f(n) \le p(n)$. In this paper, we shall use the elementary method to study the value distribution property of $f(n)$ in the following form:

$$\sum_{n\le x}(f(n) - p(n))^2,$$

where $x \ge 1$ be a real number, and give an interesting asymptotic formula for it. In fact, we shall prove the following result:

Theorem. For any real number $x \ge 3$, we have the asymptotic formula:

$$\sum_{n\le x}(f(n) - p(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Note. For any prime p and positive integer α, we define $f(p^\alpha) = p^{\frac{1}{\alpha+1}}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime power factorizations of n, then from the definition of $f(p^\alpha)$ we have

$$f(n) = \max_{1\le i\le r}\{f(p_i^{\alpha_i})\} = \max_{1\le i\le r}\left\{p_i^{\frac{1}{\alpha_i+1}}\right\}.$$

Let x be a positive real number, whether there exists an asymptotic formula for

$$\sum_{n\le x}\left(f(n) - \sqrt{p(n)}\right)^2$$

is an unsolved problem.

§2. Some lemmas

To complete the proof of the theorem, we need one simple Lemma.

Lemma. Let p be a prime and $\alpha > 0$ be an integer, then for any fixed positive integer m, we have the asymptotic formula:

$$\sum_{2\le p\le x^{\frac{1}{m}}} p^\alpha = \frac{m}{\alpha+1}\cdot\frac{x^{\frac{\alpha+1}{m}}}{\ln x} + O\left(\frac{x^{\frac{\alpha+1}{m}}}{\ln^2 x}\right).$$

Proof. Let $\pi(x)$ denotes the number of the primes up to x. Noting that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from the Abel's identity [8] we have

$$\sum_{2\le p\le x^{\frac{1}{m}}} p^{\alpha} = \pi(x^{\frac{1}{m}})\left(x^{\frac{1}{m}}\right)^{\alpha} - \alpha\int_2^{x^{\frac{1}{m}}} \pi(t)t^{\alpha-1}$$
$$= \frac{x^{\frac{\alpha+1}{m}}}{\ln x^{\frac{1}{m}}} - \frac{\alpha}{\alpha+1}\frac{x^{\frac{\alpha+1}{m}}}{\ln x^{\frac{1}{m}}} + O\left(\frac{x^{\frac{\alpha+1}{m}}}{\ln^2 x^{\frac{1}{m}}}\right) = \frac{m}{\alpha+1}\cdot\frac{x^{\frac{\alpha+1}{m}}}{\ln x} + O\left(\frac{x^{\frac{\alpha+1}{m}}}{\ln^2 x^{\frac{1}{m}}}\right).$$

This proves Lemma.

§3. Proof of the theorem

Now we completes the proof of our Theorem. For any fixed positive integer n, let $p(n)$ denotes the greatest prime divisor of n, we shall debate this problem in following three cases:

(I) If $n = n_1 p(n)$ with $(n_1,\ p(n)) = 1$ by the definition of $f(n)$, then

$$f(n) = \max\{f(n_1),\ f(p(n))\} = p(n),$$

so $f(n) - p(n) = 0$ in this case.

(II) If $n = n_1 p^2(n)$ with $(n_1,\ p(n)) = 1$, here $p(n) \le n^{\frac{1}{2}}$, hence

$$\sum_{\substack{n\le x\\ n=n_1p^2(n)}} (f(n)-p(n))^2 = \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} (f^2(n_1p^2) - 2pf(n_1p^2) + p^2)$$
$$= \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} f^2(n_1p^2) - 2\sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} f(n_1p^2)p + \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} p^2 \equiv I_1 - 2I_2 + I_3. \quad (1)$$

Let p_1 be the greatest prime divisor of n_1, if $n_1 = n_2 p_1^{\alpha}$ with $\alpha \ge 2$, then $f(n) = \sqrt{p}$; otherwise $n_1 = n_2 p_1$ with $(n_2, p_1) = 1$, so

$$I_1 = \sum_{\substack{n_1p^2\le x\\ p(n_1)<p}} f^2(n_1p^2) = \sum_{\substack{n_2p_1p^2\le x\\ p_1<p}} f^2(n_2p_1p^2) + \sum_{\substack{n_2p_1^{\alpha}p^2\le x\\ p_1<p}} p$$
$$= \sum_{\substack{n_2p_1p^2\le x\\ p_1>\sqrt{p}}} p_1^2 + \sum_{\substack{n_2p_1p^2\le x\\ p_1\le\sqrt{p}}} p + \sum_{n_2p_1^{\alpha}p^2\le x} p. \quad (2)$$

By using Lemma, we can deduce that

$$\sum_{\substack{n_2p_1p^2\le x\\ p_1>\sqrt{p}}} p_1^2 \le \sum_{n_2p_1\le x^{\frac{1}{3}}} p_1^2 \sum_{p\le\sqrt{\frac{x}{n_2p_1}}} 1 \ll \frac{x^{\frac{1}{2}}}{\ln x} \sum_{n_2p_1\le x^{\frac{1}{3}}} p_1^{\frac{3}{2}} n_1^{-\frac{1}{2}}$$
$$\ll \frac{x^{\frac{1}{2}}}{\ln x} \sum_{p_1\le x^{\frac{1}{3}}} p_1^{\frac{3}{2}} \sum_{n_2\le x^{\frac{1}{3}}/p_1} n^{-\frac{1}{2}} \ll \frac{x^{\frac{4}{3}}}{\ln^2 x}, \quad (3)$$

Similarly, we have

$$\sum_{\substack{n_2p_1p^2\le x\\ p_1<\sqrt{p}}} p \le \sum_{n_2p_1\le x^{\frac{1}{3}}} \sum_{p\le\sqrt{\frac{x}{n_2p_1}}} p \ll \frac{x}{\ln^2 x}, \quad (4)$$

$$\sum_{n_2 p_1^\alpha p^2 \le x} p \ll \frac{x}{\ln^2 x}. \tag{5}$$

From (2), (3), (4) and (5), we may immediately obtain

$$I_1 \ll \frac{x^{\frac{4}{3}}}{\ln^2 x}. \tag{6}$$

According to the estimate method of I_1, we can also get

$$\begin{aligned} I_2 &= \sum_{\substack{n_1 p^2 \le x \\ p(n_1) < p}} f(n_1 p^2) p \\ &= \sum_{\substack{n_2 p_1 p^2 \le x \\ p_1 > \sqrt{p}}} p_1 p + \sum_{\substack{n_2 p_1 p^2 \le x \\ p_1 \le \sqrt{p}}} p^{\frac{3}{2}} + \sum_{n_2 p_1^\alpha p^2 \le x} p^{\frac{3}{2}} \\ &\ll \frac{x^{\frac{7}{6}} \ln\ln x}{\ln x}. \end{aligned} \tag{7}$$

Now, we will calculate I_3. By using Lemma and note that $p(n) \le n^{\frac{1}{2}}$, we have

$$\begin{aligned} I_3 &= \sum_{\substack{n_1 p^2 \le x \\ p(n_1) < p}} p^2 \\ &= \sum_{n_1 \le x^{\frac{1}{3}}} \sum_{p^2 \le x/n_1} p^2 \\ &= \sum_{n_1 \le x^{\frac{1}{3}}} \left(\frac{2x^{\frac{3}{2}}}{3n_1^{\frac{3}{2}} (\ln x - \ln m)} + O\left(\frac{x^{\frac{3}{2}}}{n_1^{\frac{3}{2}} \ln^2 \sqrt{\frac{x}{n_1}}} \right) \right) \\ &= \frac{2x^{\frac{3}{2}}}{3\ln x} \sum_{n_1 \le e^{\sqrt{\ln x}}} \frac{1}{n_1^{\frac{3}{2}}} + O\left(\sum_{e^{\sqrt{\ln x}} < n_1 \le x^{\frac{1}{3}}} \frac{x^{\frac{3}{2}}}{n_1^{\frac{3}{2}} \ln \frac{x}{n_1}} \right) + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right) \\ &= \frac{2\zeta(\frac{3}{2}) x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right), \end{aligned} \tag{8}$$

where $\zeta(s)$ is the Riemann zeta-function.

From (1), (6), (7) and (8), we may immediately deduce the case (II)

$$\sum_{n \le x} (f(n) - p(n))^2 = \frac{2\zeta(\frac{3}{2}) x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x} \right).$$

(III) If $n = n_1 p^\alpha(n)$ with $\alpha \ge 3$, note that

$$\begin{aligned} \sum_{\substack{n_1 p^\alpha \le x \\ p(n_1) < p}} p^2 &= \sum_{n_1 \le x} \sum_{p \le \left(\frac{x}{n_1}\right)^{\frac{1}{\alpha}}} p^2 \\ &\ll \frac{x^{\frac{3}{\alpha}}}{\ln x} \sum_{n_1 \le x} \left(\frac{1}{n}\right)^{-\frac{3}{\alpha}} \ll \frac{x^{\frac{3}{2}}}{\ln^2 x}, \end{aligned}$$

so in this case

$$\sum_{n\leq x}(f(n)-p(n))^2 \ll \frac{x^{\frac{3}{2}}}{\ln^2 x}.$$

Combining three cases above, for any real number $x \geq 3$, we have the asymptotic formula:

$$\sum_{n\leq x}(f(n)-p(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

This completes the proof of Theorem.

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