Scientia Magna
Vol. 4 (2008), No. 2, 29-33

On the solvability of an equation involving the Smarandache function and Euler function

Weiguo Duan[†‡] and Yanrong Xue[†]

† Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R.China
‡ Department of Mathematics, Weinan Teacher's College, Weinan, Shaanxi, P.R.China

Abstract For any positive integer n, let $\phi(n)$ and $S(n)$ be the Euler function and the Smarandache function respectively. In this paper, we use the properties and the curve figure of these two functions to study the solvability of the equation $\sum_{i=1}^{n} S(i) = \phi(\frac{n(n+1)}{2})$, and prove that this equation has only two positive integer solutions $n = 1, 10$.

Keywords Euler function, F. Smarandache function, equation, solvability.

§1. Introduction and result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$, where N denotes the set of all positive integers. From the definition of $S(n)$, it is easy to see that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then we have

$$S(n) = \max_{1 \le i \le k} \{S(p_i^{\alpha_i})\}.$$

It is clear that from this properties we can calculate the value of $S(n)$, the first few values of $S(n)$ are: $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots$. About the arithmetical properties of $S(n)$, some authors had studied it, and obtained many interesting results. For example, Lu Yaming [2] studied the solvability of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \ge 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1, \ m_2, \ \cdots, \ m_k)$.

Jozsef Sandor [3] proved that for any positive integer $k \ge 2$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ satisfying the inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

Rongji Chen [5] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any fixed $r \in N$ with $r \geq 3$, the positive integer n is a solution of

$$S(n)^r + S(n)^{r-1} + \cdots + S(n) = n$$

if and only if

$$n = p(p^{r-1} + p^{r-2} + \cdots + 1),$$

where p is an odd prime satisfying $p^{r-1} + p^{r-2} + \cdots + 1 | (p-1)!$.

Xiaoyan Li and Yanrong Xue [6] proved that for any positive integer k, the equation $S(n)^2 + S(n) = kn$ has infinite positive integer solutions, and each solution n has the form $n = pn_1$, where $p = kn_1 - 1$ is a prime.

For any positive integer n, the Euler function $\phi(n)$ is defined as the number of all positive integers not exceeding n, which are relatively prime to n. It is clear that $\phi(n)$ is a multiplicative function.

In this paper, we shall use the elementary method and compiler program to study the solvability of the equation:

$$S(1) + S(2) + \cdots + S(n) = \phi\left(\frac{n(n+1)}{2}\right), \tag{1}$$

and give its all positive integer solutions. That is, we shall prove the following:

Theorem. The equation

$$S(1) + S(2) + \cdots + S(n) = \phi\left(\frac{n(n+1)}{2}\right)$$

has and only has two positive integer solutions $n = 1,\ 10$.

§2. Main lemmas

In this section, we shall give two simple lemmas which are necessary in the proof of our Theorem. First we have the following:

Lemma 1. For any positive integer $n > 100$, we have the inequality

$$\sum_{i=1}^{n} S(i) \leqslant \frac{\pi^2}{11.99} \cdot \frac{n^2}{\ln n}.$$

Proof. From the mean value formula of $S(n)$ (See reference [7])

$$\sum_{n \leqslant x} S(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \mathrm{O}\left(\frac{x^2}{\ln^2 x}\right)$$

we know that there exists one constant $N > 0$ such that

$$\sum_{i=1}^{n} S(i) \leqslant \frac{\pi^2}{12} \cdot \frac{n^2}{\ln n} + \frac{1}{1199} \cdot \frac{\pi^2}{12} \cdot \frac{n^2}{\ln n} \leqslant \frac{\pi^2}{11.99} \cdot \frac{n^2}{\ln n}$$

holds for all positive integer $n > N$. We can take $N = 100$ by calculation. This completes the proof of Lemma 1.

Lemma 2. For Euler function $\phi(n)$, we have the estimate

$$\phi\left(\frac{n(n+1)}{2}\right) > \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot \frac{1}{\ln^{1.5}\left(2\ln\frac{n(n+1)}{2}\right)}.$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then there always exist some primes $p_1,\ p_2,\ \cdots p_s$ such that $p_1 p_2 \cdots p_s > n$. From [1] we have

$$\sum_{p \leqslant x} \ln p = x + \mathrm{O}\left(\frac{x}{\log x}\right),$$

by this estimate we know that

$$\ln n < \sum_{i=1}^{s} \ln p_i \leqslant \sum_{p_i \leqslant p_s} \ln p_i \leqslant p_s < 2\ln n.$$

Thus

$$\sum_{p|n} \frac{1}{p} \leqslant \sum_{p_i \leqslant p_s} \frac{1}{p_i} \leqslant \ln\ln p_s < \ln\ln(2\ln n).$$

Note that $\phi(n) = n\prod\limits_{p|n}\left(1-\frac{1}{p}\right)$, if $\frac{n(n+1)}{2}$ is even, then

$$\begin{aligned}
\phi\left(\frac{n(n+1)}{2}\right) &= \frac{n(n+1)}{2}\prod_{p|\frac{n(n+1)}{2}}\left(1-\frac{1}{p}\right) \\
&= \frac{n(n+1)}{4} e^{\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2} \ln(1-\frac{1}{p})+\frac{1.5}{p}-\frac{1.5}{p}} \\
&= \frac{n(n+1)}{4} e^{-\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2}\frac{1.5}{p}+\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2}[\ln(1-\frac{1}{p})+\frac{1.5}{p}]} \\
&= \frac{n(n+1)}{4} e^{-\sum\limits_{p|\frac{n(n+1)}{2},p\neq 2}\frac{1.5}{p}} \\
&\geqslant \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot e^{-1.5\ln\ln(2\ln\frac{n(n+1)}{2})} \\
&> \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot \frac{1}{\ln^{1.5}\left(2\ln\frac{n(n+1)}{2}\right)}.
\end{aligned}$$

If $\frac{n(n+1)}{2}$ is odd, we can also get the same result. This completes the proof of Lemma 1.

§3. Proof of the theorem

In this section, we shall complete the proof of our Theorem. First we study the tendency of the functional digraph

$$f(x) = \frac{x(x+1)}{4} \cdot e^{\frac{3}{4}} \frac{1}{\ln^{1.5}(2\ln\frac{x(x+1)}{2})} - \frac{\pi^2}{11.99} \cdot \frac{x^2}{\ln x}.$$

By use of Mathematica compiler program we find that $f(x) > 0$, if $x > 100754$.

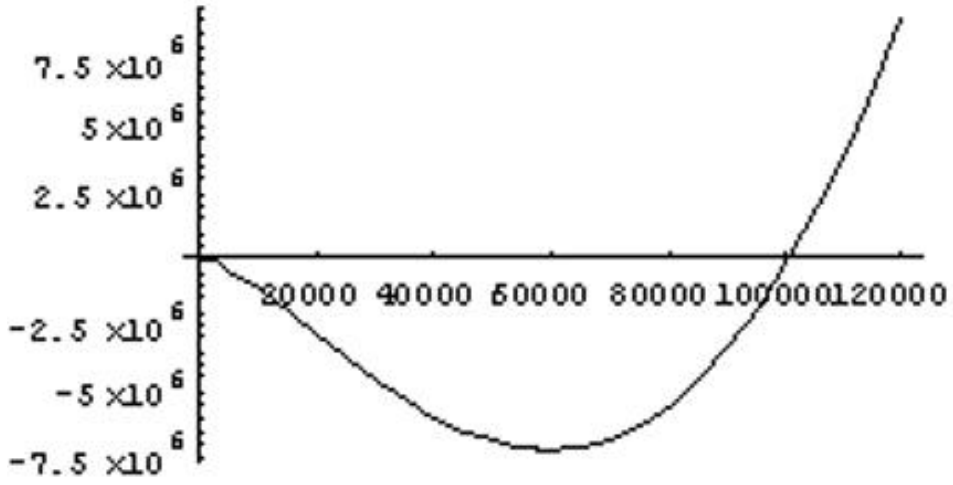

figure 1

From the figure 1 we know that if $n > 100754$, then

$$\sum_{i=1}^{n} S(i) \leqslant \frac{\pi^2}{11.99} \cdot \frac{n^2}{\ln n} < \frac{n(n+1)}{4} \cdot e^{\frac{3}{4}} \cdot \frac{1}{\ln^{1.5}(2\ln\frac{n(n+1)}{2})} < \phi\left(\frac{n(n+1)}{2}\right). \tag{2}$$

If $x \in (100754, +\infty)$, we use Mathematica compiler program to compute $f^{'}(x)$, then we find that the derivative $f^{'}(x)$ is positive, so (2) is also true if $x > 100754$.

Now we consider the solution of (1) for all $n \in [1,\ 100754]$. By use of the computer programming language, we obtain that the equation (1) has no any other positive integer solutions except $n = 1,\ n = 10$. This completes the proof of Theorem.

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[2] Lu Yaming, On the solutions of an equation involving the Smarandache function, Scientia Magna, **2**(2006), No. 1, 76-79.

[3] Jozsef Sandor, On certain inequalities involving the Smarandache function, Scientia Magna, **2**(2006), No. 3, 78-80.

[4] Pan C. D. and Pan C. B., Elementary Number Theory, Beijing University Press, Beijing, 1992.

[5] Rongji Chen, On the functional equation $S(n)^r+S(n)^{r-1}+\cdots+S(n) = n$, Smaraandache Notions Journal, **11**(2000), No. 1-2-3, 128-130.

[6] Xiaoyan Li and Yanrong Xue, On an equation related to function $S(n)$, Scientia Magna, **4**(2008), No. 1, 148-151.

[7] Y. X. Wang, On the Smarandache function, Research on Smarandache problems in number theory, II (2005), 103-106.

[8] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.

The computing programme is given as follows if $n \in [1, 100754]$.

```
# include "stdio.h"
# include "math.h"
# define N 100754
int S(int n)
{int ret=1,num=n;
unsigned long int nn=1;
for(ret=1;ret<=n;ret++) { nn=nn*ret;
if(nn%num==0) break;} if (ret>n) ret=n;
return ret; }
int SumS(int n)
{int ret=0,i;
for(i=1;i<=n;i++) ret+=S(i);
return ret;}
int coprime(int i,int n)
{ int a=n,b=i;
while(a!=b) { if(a==0) return b;
if(b==0) return a;
if(a>b) a=a%b;
else
b=b%a;}
return a; }
int Euler(int n)
{int ret=1,i;
for(i=2;i<n;i++) {if(coprime(i,n)==1) ret++;} return ret;}
main()
{ int kk;
for(kk=1;kk<=N;kk++) if(SumS(kk)==Euler((kk*(kk+1)/2)))
printf( "rusult is % d\n" ,kk);
getch (); }
```

Scientia Magna
Vol. 4 (2008), No. 2, 34-38

The generalized Hermitian Procrustes problem and its approximation

Maolin Liang, Wansheng He and Lifang Dai

College of Mathematics and Statistics, Tianshui Normal University
Tianshui, Gansu, 741001, P. R. China
liangml2005@163.com

Abstract For orthogonal projective matrix R, i.e., $R^2 = R$ and $R^T = R$, we say that A is generalized Hermitian matrix, if $RAR = A^*$. In this paper, we investigate the least residual problem $\|AX - B\| = \min$ with given X, B, and associated optimal approximation problem in the generalized Hermitian matrix set. The general expressions of the solutions are derived by matrix decomposition.

Keywords Generalized Hermitian matrix, full-rank factorization, Procrustes problem, optimal approximation.

§1. Introduction

Some symbols and notations: Let $C_r^{m\times n}$ be the set of all $m\times n$ complex matrices with rank r, $HC^{n\times n}$ be the set of all $n\times n$ Hermitian matrices. Denoted by A^+, A^*, $rank(A)$ the Moore-Penrose generalized inverse, conjugate transpose, rank of matrix A, respectively. Moreover, I_n represents identity matrix of order n, and $J = (e_n, e_{n-1}, \cdots, e_1)$, $e_i \in C^n$ is the ith column of I_n. $\|\cdot\|$ stands for the Frobenius norm. Matrix $R \in C_r^{n\times n}$ is said to be projective (orthogonal projective) matrix, if $R^2 = R$ ($R^2 = R$ and $R^* = R$).

Definition 1.1. If $A \in C^{n\times n}$, we say that A is centro-symmetric matrix, if $JAJ = A$.

The centro-symmetric matrix has important and practical applications in information theory, linear system theory and numerical analysis (see [1-2]). As the extension of the centro-symmetric matrix, we define the following conception.

Definition 1.2. For given orthogonal projective matrix $R \in C_r^{n\times n}$, we say that $A \in C^{n\times n}$ is generalized Hermitian matrix, if $RAR = A^*$. Denote the set of all generalized Hermitian matrices by $GHC^{n\times n}$.

In this paper, we discuss two problems as follows:

Problem I.(Procrustes Problem): Given orthogonal projective matrix $R \in R^{n\times n}$, and $X, B \in C^{n\times m}$, find $A \in GHC^{n\times n}$ such that

$$\|AX - B\| = \min.$$

Problem II.(Optimal Approximation Problem): Given $M \in C^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\| M - \hat{A}\| = \min_{A \in S_E} \|M - A\|,$$

where S_E is the solution set of Problem I.

Obviously, when $M = 0$, Problem II is changed into finding the least Frobenius norm solution of Problem I.

Many important results have been achieved about the above problems with different matrix sets, such as centro-symmetric matrix[3], symmetric matrix[4−5], R-symmetric matrix[6−7] and (R,S)-symmetric matrix[8] set. In this paper, we investigate the above problems in the generalized Hermitian matrix set by matrix decomposition.

§2. Preliminary knowledge

In this section, we discuss the properties and structures of (orthogonal) projective matrices $R \in C_r^{n\times n}$ and $A \in GHC^{n\times n}$.

Denote $s = rank(I-R)$, we know that $r+s = n$ since $R^2 = R$. Suppose that $p_1, p_2, \ldots, p_r$ and $q_1, q_2, \ldots, q_s$ are the normal orthogonal basis for range $\mathbf{R}(R)$ and null space $\mathbf{N}(R)$ of R, respectively. Let $P = (p_1, p_2, \ldots, p_r) \in C_r^{n\times r}$ and $Q = (q_1, q_2, \ldots, q_s) \in C_s^{n\times s}$, then

$$P^*P = I_r,\ Q^*Q = I_s, \tag{1}$$

$$RP = P,\ RQ = 0. \tag{2}$$

Lemma 2.1.(see [9]) Let matrix $A \in C_r^{n\times m}$ and its full-rank factorization $A = FG$, where $F \in C_r^{n\times r}$, $G \in C_r^{r\times m}$, then A is projective matrix if and only if $GF = I_r$.

Lemma 2.2. $R \in C_r^{n\times n}$ is projective matrix, then

$$R = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix}, \tag{3}$$

where matrix $\begin{pmatrix} P & Q \end{pmatrix}$ is invertible, and $\begin{pmatrix} P & Q \end{pmatrix}^{-1} = \begin{pmatrix} \widehat{P} \\ \widehat{Q} \end{pmatrix}$.

If R is orthogonal projective matrix, we have

$$R = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \tag{4}$$

where $\begin{pmatrix} P & Q \end{pmatrix}$ is unitary matrix.

Proof. Assume that the full-rank factorization of R is $R = P\widehat{P}$, we obtain from Lemma 2.1 and (1) that

$$\widehat{P} = P^*R,\ \widehat{P}P = I_r. \tag{5}$$

Similarly, if the full-rank factorization of $I - R$ is $I - R = Q\widehat{Q}$, we generate

$$\widehat{Q} = Q^*(I - R),\ \widehat{Q}Q = I_s, \tag{6}$$

since $(I - R)^2 = I - R$. Connecting with (1)(2)(5) and (6), we know that (3) holds. The equality (4) is obvious since $R^* = R$.

Lemma 2.3. Given matrices R as in (4) and $A \in GHC^{n\times n}$, then

$$A = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \forall\, G \in HC^{r\times r}. \tag{7}$$

Proof. According to Lemma 2.2 and Definition 2.1, it is clear that (7) holds.

Lemma 2.3 indicates that arbitrary matrix $M \in C^{n\times n}$ can be written as

$$M = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}.$$

§3. The solutions of Problem I and II

Given matrices $X, B \in C^{n\times m}$, partition

$$\begin{pmatrix} P^* \\ Q^* \end{pmatrix} X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } \begin{pmatrix} P^* \\ Q^* \end{pmatrix} B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{8}$$

where $X_1, B_1 \in C^{r\times m}$ and $X_2, B_2 \in C^{s\times m}$.

We need the following two lemmas derived from References [7] and [8], respectively.

Lemma 3.1. Suppose that matrices X_1, B_1 in (8), then matrix equation $A_1X_1 = B_1$ is consistent for $A_1 \in HC^{r\times r}$, if and only if $B_1X_1^+X_1 = B_1$ and $X_1^*B_1 = B_1^*X_1$, the general solution is

$$A_1 = \tilde{A}_1 + (I_r - X_1X_1^+)K_1(I_r - X_1X_1^+),$$

where $\tilde{A}_1 = (I_r - \frac{X_1X_1^+}{2})B_1X_1^+ + (B_1X_1^+)^*(I_r - \frac{X_1X_1^+}{2}), \forall\, K_1 \in HC^{r\times r}$.

Lemma 3.2. Given matrices X_1, B_1 in (8), then

$$\min_{G\in C^{r\times r}} \parallel GX_1 - B_1 \parallel = \parallel B_1(I_r - X_1^+X_1) \parallel$$

if and only if $G = B_1X_1^+ + K_2(I_r - X_1X_1^+), \forall\, K_2 \in C^{r\times r}$.

According to Lemmas 3.1 and 3.2, we obtain

Lemma 3.3. For the above given matrices X_1, B_1,

$$\min_{A_1\in HC^{r\times r}} \parallel A_1X_1 - B_1 \parallel = \parallel B_1(I_r - X_1^+X_1) \parallel$$

if and only if

$$X_1^*B_1X_1^+ = X_1^+X_1B_1^*X_1X_1^+, \tag{9}$$

and the expression of A_1 is the same as that in Lemma 3.1.

Proof. $\parallel A_1X_1 - B_1 \parallel^2 = \parallel B_1 - B_1X_1^+X_1 + B_1X_1^+X_1 - A_1X_1 \parallel^2$

$$= \parallel B_1(I_r - X_1^+X_1) \parallel^2 + \parallel B_1X_1^+X_1 - A_1X_1 \parallel^2$$

Hence, the least residual can be attained only if $B_1X_1^+X_1 = A_1X_1$, which is consistent for $A_1 \in HC^{r\times r}$ under condition (9) by Lemma 3.3. The proof is completed.

Based on the previous analysis, Problem I can be solved in the following Theorem.

Theorem 3.1. Given matrix R as in (4), $X, B \in C^{n\times m}$ and the partition (8), then

$$\min_{A\in GHC^{n\times n}} \|AX - B\|^2 = \|B_1(I_r - X_1^+X_1)\|^2 + \|B_2\|^2, \tag{10}$$

if and only if (9) holds, at this time

$$A = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} \widehat{G} + (I_r - X_1X_1^+)K(I_r - X_1X_1^+) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \tag{11}$$

where $\tilde{G} = (I_r - \frac{X_1X_1^+}{2})B_1X_1^+ + (B_1X_1^+)^*(I_r - \frac{X_1X_1^+}{2})$, $\forall\, K \in HC^{r\times r}$.

Proof. According to the unitary invariance of Frobenius norm, formulas (4) and (7), we obtain

$$\begin{aligned} &\|AX - B\|^2 \\ &= \left\| \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix} X - B \right\|^2 \\ &= \left\| \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right\|^2 \\ &= \|GX_1 - B_1\|^2 + \|B_2\|^2. \end{aligned}$$

Therefore, the problem (10) is equivalent to the following least residual problem

$$\min_{G\in HC^{r\times r}} \| GX_1 - B_1 \| .$$

From Lemma 3.3, we know that the minimum can be attained if and only if (9), and

$$G = \tilde{G} + (I_r - X_1X_1^+)K(I_r - X_1X_1^+),$$

where $K \in HC^{r\times r}$ is arbitrary. Submitting G into (7), then (11) holds.

The following lemma stated from [6].

Lemma 3.4. Let $L \in C^{q\times m}, \Delta \in C^{q\times q}$, $\Gamma \in C^{m\times m}$, and $\Delta^2 = \Delta = \Delta^*$, $\Gamma^2 = \Gamma = \Gamma^*$, then $\| L - \Delta L\Gamma \| = \min\limits_{N\in C^{q\times m}} \| L - \Delta N\Gamma \|$ if and only if $\Delta(L - N)\Gamma = 0$.

Let S_E be the solution set of Problem I. We can easily verify from its definition that S_E is a closed convex subsets in matrix space $C^{n\times n}$ under Frobenius norm. The optimal approximation theorem[10] reveals that Problem II has unique solution, which can be expressed in the next theorem.

Theorem 3.2. Suppose that the given matrix in Problem II is

$$M = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix} \in C^{n\times n}$$

then

$$\min_{A\in S_E} \| M - A \| \tag{12}$$

if and only if

$$A = \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} \widehat{G} + (I_r - X_1X_1^+)\frac{M_1+M_1^*}{2}(I_r - X_1X_1^+) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^* \\ Q^* \end{pmatrix}, \tag{13}$$

where $\widehat{G}$ is the same as that in Theorem 3.1.

Proof. By using the unitary invariance of Frobenius norm and Theorem 3.1, we obtain

$$\begin{aligned} \| M - A \|^2 &= \left\| \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} - \begin{pmatrix} \widehat{G} + (I_r - X_1X_1^+)K(I_r - X_1X_1^+) & 0 \\ 0 & 0 \end{pmatrix} \right\|^2 \\ &= \| (M_1 - \widehat{G}) - (I_r - X_1X_1^+)K(I_r - X_1X_1^+) \|^2 \\ &\quad + \| M_2 \|^2 + \| M_3 \|^2 + \| M_4 \|^2, \end{aligned}$$

then the problem (12) equals to solve the minimum problem

$$\min_{K\in HC^{r\times r}} \| (M_1 - \widehat{G}) - (I_r - X_1X_1^+)K(I_r - X_1X_1^+) \| .$$

Moreover, since $\|M_1\|^2 = \|\frac{M_1+M_1^*}{2}\|^2 + \|\frac{M_1-M_1^*}{2}\|^2$, hence the above minimum problem can be transformed equivalently as

$$\min_{K\in HC^{r\times r}} \| (\frac{M_1 + M_1^*}{2} - \widehat{G_1}) - (I_r - X_1X_1^+)K(I_r - X_1X_1^+) \| .$$

We further deduce from Lemma 3.4 that

$$(I_r - X_1X_1^+)K(I_r - X_1X_1^+) = (I_r - X_1X_1^+)\frac{M_1+M_1^*}{2}(I_r - X_1X_1^+), \tag{14}$$

submitting (14) into (11), we obtain (13).

References

[1] A. L. Andrew, Solution of equations involving centrosymmetric matrices, Technometrics, **15**(1973), 405-407.

[2] F. Z. Zhou, X. Y. Hu, L. Zhang, The solvability conditions for the inverse eigenvalue problems of centro-symmetric matrices, Linear Algebra Appl., **364**(2003), 147-160.

[3] F. Z. Zhou, X. Y. Hu and L. Zhang, Least-squares solutions for inverse problem of centro-symmetric matrices, Comput. Math Appl., **45**(2003), 1581-1589.

[4] K. E. Chu, Symmetric solutions of linear matrix equations by matrix decompositions, Linear Algebra Appl., **119**(1989), 35-50.

[5] N. N. Higham, The symmetric Procrustes problem, BIT, **28**(1988), 133-143.

[6] W. F. Trench, Inverse eigenproblems and associated approximation problems for matrices with generalized symmetry or skew symmetry, Linear Algebra Appl., **380**(2004), 199-211.

[7] W. F. Trench, Hermitian,Hermitian R-symmmetrc, and Hermitian R-skew symmetric Procrustes problems, Linear Algebra Appl., **387**(2004), 83-98.

[8] W. F. Trench, Minimization problems for (R,S)-symmetrric and (R,S)-skew symmetric matrices, Linear Algebra Appl., **389**(2004), 23-31.

[9] G. R. Wang, Y. M. Wei and S. Z. Qiao, Generalized Inverse, Theory and Computations, Beijing Science Press, 2004.

[10] Q. X. Cheng, D. Z. Zhang, Real Variable Function and Functional Analysis, Advanced Educational Press, 1983.