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# Some identities on $k$-power complement

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**Abstract** The main purpose of this paper is to calculate the value of the series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^\beta(n)},$$

where $a_k(n)$ is the $k$-power complement number of any positive number $n$, and $\alpha$, $\beta$ are two complex numbers with $Re(\alpha) \geq 1$, $Re(\beta) \geq 1$. Several interesting identities are given.

**Keywords** $k$-power complement number, identities, Riemann zeta-function.

## §1. Introduction

For any given natural number $k \geq 2$ and any positive integer $n$, we call $a_k(n)$ as a $k$-power complement number if $a_k(n)$ denotes the smallest positive integer such that $n \cdot a_k(n)$ is a perfect $k$-power. Especially, we call $a_2(n), a_3(n), a_4(n)$ as the square complement number, cubic complement number, quartic complement number respectively. In reference [1], Professor F.Smarandache asked us to study the properties of the $k$-power complement number sequence. About this problem, there are many authors had studied it, and obtained many results. For example, in reference [2], Professor Wenpeng Zhang calculated the value of the series

$$\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_k(n))^s},$$

where $s$ is a complex number with $Re(\alpha) \geq 1$, $k$=2, 3, 4. Maohua Le [3] discussed the convergence of the series

$$s_1 = \sum_{n=1}^{+\infty} \frac{1}{a_2^m(n)}$$

and

$$s_2 = \sum_{n=2}^{+\infty} \frac{(-1)^n}{a_2(n)},$$

where $m \leq 1$ is a positive number, and proved that they are both divergence.

But about the properties of the $k$-power complement number, we still know very little at present. This paper, as a note of [2], we shall give a general calculate formula for

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^\beta(n)}.$$

That is, we shall prove the following:

**Theorem 1.** For any complex numbers $\alpha$, $\beta$ with $\mathtt{Re}(\alpha) \geq 1$, $\mathtt{Re}(\beta) \geq 1$, we have

$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha \cdot a_k^\beta(n)} = \zeta(k\alpha) \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right),$$

where $\zeta(\alpha)$ is the Riemann zeta-function, $\prod_p$ denotes the product over all prime $p$.

**Theorem 2.** For any complex numbers $\alpha$, $\beta$ with $\mathtt{Re}(\alpha) \geq 1$, $\mathtt{Re}(\beta) \geq 1$, we have

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^\alpha \cdot a_k^\beta(n)} = \left(1 - \frac{2(2^{k\alpha} - 1)(2^{\alpha+(k+1)\beta} - 1)}{2^{(k+1)\alpha+(k-1)\beta} - 2^{\alpha-(k-1)^2\beta}}\right) \zeta(k\alpha) \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right).$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(8) = \frac{\pi^8}{9450}$. From our Theorems we may immediately obtain the following two corollaries:

**Corollary 1.** Taking $\alpha = \beta$, $k = 2$ in above Theorems, then we have

$$\sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^\alpha} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)};$$

$$\sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{(n \cdot a_2(n))^\alpha} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)} \cdot \frac{4^\alpha - 1}{4^\alpha + 1};$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{(n \cdot a_2(n))^\alpha} = \frac{\zeta^2(2\alpha)}{\zeta(4\alpha)} \cdot \frac{3 - 4^\alpha}{1 + 4^\alpha}.$$

**Corollary 2.** Taking $\alpha = \beta = 1$, $2$, $k = 2$ in Corollary 1, we have

$$\sum_{n=1}^{+\infty} \frac{1}{n \cdot a_2(n)} = \frac{5}{2}, \quad \sum_{n=1}^{+\infty} \frac{1}{(n \cdot a_2(n))^2} = \frac{7}{6};$$

$$\sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{n \cdot a_2(n)} = \frac{3}{2}, \quad \sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{(n \cdot a_2(n))^2} = \frac{35}{34};$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n \cdot a_2(n)} = -\frac{1}{2}, \quad \sum_{n=1}^{+\infty} \frac{(-1)^n}{(n \cdot a_2(n))^2} = -\frac{91}{102}.$$

## §2. Proof of the theorem

In this section, we will complete the proof of the theorems. For any positive integer $n$, we can write it as $n = m^k \cdot l$, where $l$ is a $k$-free number, then from the definition of $a_k(n)$ we have

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{n^{\alpha} \cdot a_k^{\beta}(n)} &= \sum_{m=1}^{+\infty} \sum_{l=1}^{+\infty} \frac{\sum\limits_{d^k|l} \mu(d)}{m^{k\alpha} l^{\alpha} l^{(k-1)\beta}} \\
&= \zeta(k\alpha) \sum_{l=1}^{+\infty} \frac{\sum\limits_{d^k|l} \mu(d)}{l^{\alpha+(k-1)\beta}} \\
&= \zeta(k\alpha) \prod_p \left(1 + \frac{1}{p^{\alpha+(k-1)\beta}} + \frac{1}{p^{2(\alpha+(k-1)\beta)}} + \cdots + \frac{1}{p^{(k-1)(\alpha+(k-1)\beta)}}\right) \\
&= \zeta(k\alpha) \prod_p \left(1 + \frac{1}{p^{\alpha+(k-1)\beta}} \frac{1 - \frac{1}{p^{(k-1)(\alpha+(k-1)\beta)}}}{1 - \frac{1}{p^{\alpha+(k-1)\beta}}}\right) \\
&= \zeta(k\alpha) \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right),
\end{aligned}$$

where $\mu(n)$ denotes the Möbius function. This completes the proof of Theorem 1.

Now we come to prove Theorem 2. First we shall prove the following identity

$$\begin{aligned}
\sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{n^{\alpha} \cdot a_k^{\beta}(n)} &= \sum_{\substack{m=1 \\ 2\nmid m^k l}}^{+\infty} \sum_{l=1}^{+\infty} \frac{\sum\limits_{d^k|l} \mu(d)}{m^{k\alpha} l^{\alpha} l^{(k-1)}} \\
&= \sum_{\substack{m=1 \\ 2\nmid m}}^{+\infty} \frac{1}{m^{k\alpha}} \sum_{\substack{l=1 \\ 2\nmid l}}^{+\infty} \frac{\sum\limits_{d^k|l} \mu(d)}{l^{\alpha+(k-1)}} \\
&= \frac{2^{k\alpha} - 1}{2^{k\alpha}} \cdot \frac{\zeta(k\alpha)(2^{\alpha+(k-1)\beta} - 1)}{2^{\alpha+(k-1)\beta} - 2^{(k-1)(\alpha+(k-1)\beta)}} \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right) \\
&= \frac{\zeta(k\alpha)(2^{k^{\alpha}} - 1)(2^{\alpha+(k-1)\beta})}{2^{(k+1)\alpha+(k-1)\beta} - 2^{\alpha-(k-1)^2\beta}} \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right).
\end{aligned}$$

Then use this identity and Theorem 1 we have

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^{\alpha} \cdot a_k^{\beta}(n)} \\
= \;& \sum_{n=1}^{+\infty} \frac{1}{n^{\alpha} \cdot a_k^{\beta}(n)} - 2 \sum_{\substack{n=1 \\ 2\nmid n}}^{+\infty} \frac{1}{n^{\alpha} \cdot a_k^{\beta}(n)} \\
= \;& \left(1 - \frac{2(2^{k\alpha} - 1)(2^{\alpha+(k-1)\beta} - 1)}{2^{(k+1)\alpha+(k-1)\beta} - 2^{(k-1)^2\beta-\alpha}}\right) \zeta(k\alpha) \prod_p \left(1 + \frac{1 - \frac{1}{p^{(k-1)\alpha+(k-1)^2\beta}}}{p^{\alpha+(k-1)\beta} - 1}\right).
\end{aligned}$$

This completes the proof of Theorem 2.

## References

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[3] Maohua Le, Some Problems Concerning the Smarandache Square Complementary Function, Smarandache Notions Journal, **14**(2004), 220-222.