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The Number of Spanning Trees in Generalized Complete Multipartite Graphs of Fan-Type

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Abstract: Let $K_{k,\overline{n}}$ be a complete k-partite graph of order n and let $K^{\mathscr{F}}_{k,\overline{n}}$ be a generalized complete k-partite graph of order n spanned by the fan set $\mathscr{F} = \{F_{n_1}, F_{n_2}, \cdots, F_{n_k}\}$, where $\overline{n} = \{n_1, n_2, \cdots, n_k\}$ and $n = n_1 + n_2 + \cdots + n_k$ for $1 \leqslant k \leqslant n$. In this paper, we get the number of spanning trees in $K_{k,\overline{n}}$ to be

$$t(K_{k,\overline{n}}) = n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}.$$

and the number of spanning trees in $K^{\mathscr{F}}_{k,\overline{n}}$ to be

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2}\prod_{i=1}^{k}\frac{\alpha_i^{n_i-1}-\beta_i^{n_i-1}}{\alpha_i-\beta_i}$$

where $\alpha_i = (d_i + \sqrt{d_i^2-4})/2$ and $\beta_i = (d_i - \sqrt{d_i^2-4})/2, d_i = n - n_i + 3$. In particular, $K_{1,\overline{n}} = K_n^c$ with $t(K_{1,\overline{n}}) = 0$, $K_{n,\overline{n}} = K_n$ with $t(K_{n,\overline{n}}) = n^{n-2}$ which is just the Cayley's formula and $K^{\mathscr{F}}_{1,\overline{n}} = F_n$ with $t(K^{\mathscr{F}}_{1,\overline{n}}) = (\alpha^{n-1} - \beta^{n-1})/\sqrt{5}$ where $\alpha = (3+\sqrt{5})/2$ and $\beta = (3-\sqrt{5})/2$ which is just the formula given by Z.R.Bogdanowicz in 2008.

Key Words: Connected simple graph, k-partite graph, complete graph, tree, Smarandache (E_1, E_2)-number of trees.

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§1. Introduction

Graphs considered here are simple finite and undirected. A graph is *simple* if it contains neither multiple edges nor loops. A graph is denoted by $G = \langle V(G), E(G)\rangle$ with n vertices and m edges where $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{e_1, e_2, \cdots, e_m\}$ denote the sets of its vertices and

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edges, respectively. The *degree* of a vertex v in a graph G is the number of edges incident with v and is denoted by $d(v) = d_G(v)$.

For simple graphs $G_i = \langle V_i(G), E_i(G)\rangle$ with vertex set $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, the empty graphs of order n_i are denoted by $N_{n_i} = \langle V_i, \phi\rangle$, $i = 1, 2, \cdots, k$. A complete k-partite graph $K_{k,\overline{n}} = K_{n_1,n_2,\cdots,n_k} = \langle V_1, V_2, \cdots, V_k, E\rangle$ is said to be one spanned by the empty graph set $\mathscr{N} = \{N_{n_1}, N_{n_2}, \cdots, N_{n_k}\}$, denoted by $K^{\mathscr{N}}_{n_1,n_2,\cdots,n_k}$or $K^{\mathscr{N}}_{k,\overline{n}}$ where $\overline{n} = \{n_1, n_2, \cdots, n_k\}$.

In generally, for the graph set $\mathscr{G} = \{G_{n_1}, G_{n_2}, \cdots, G_{n_k}\}$, the graph

$$K^{\mathscr{G}}_{k,\overline{n}} = K^{\mathscr{N}}_{k,\overline{n}} \cup G_{n_1} \cup G_{n_2} \cup \cdots \cup G_{n_k} \tag{1}$$

is said to be *a generalized complete k-partite graph* spanned by the graph set $\mathscr{G}$.

For all graph theoretic terminology not described here we refer to [1]. Let G be a connected graph and $E_1, E_2 \subset E(E)$ with $E_1 \neq E_2$. The *Smarandache* (E_1, E_2)*-number* $t^S(E_1, E_2)$ of trees is the number of such spanning trees T in G with $E(T) \cap E_1 \neq \emptyset$ but $E(T) \cap E_2 = \emptyset$. Particularly, if $E_1 = E(G)$ and $E_2 = \emptyset$, i.e., such number is just the number of labeled spanning trees of a graph G, denoted by $t(G)$. For a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees. One of the first such results is due to Cayley [2] who showed in 1889 that complete graph on n vertices, K_n, has n^{n-2} spanning trees. That is

$$t(K_n) = n^{n-2} \quad \text{for} \quad n \geqslant 2. \tag{2}$$

Another result is due to Sedlacek [3] who derived in 1970 a formula for the wheel on $n+1$ vertices, W_{n+1}, which is formed from a cycle C_n on n vertices by adding a new vertex adjacent to every vertex of C_n. That is

$$t(W_{n+1}) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2 \quad \text{for} \quad n \geqslant 3. \tag{3}$$

Sedlacek [4] also later derived a formula for the number of spanning trees in a Möbius ladder, M_n, is formed from a cycle C_{2n} on $2n$ vertices ladder $v_1, v_2, \cdots, v_{2n}$ by adding edges $v_i v_{n+i}$ for every vertex v_i, where $i \leqslant n$. That is

$$t(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2] \quad \text{for} \quad n \geqslant 2. \tag{4}$$

In 1985, Baron et al [5] derived the formula for the number of spanning trees in a square of cycle, C_n^2, which is expressed as follows.

$$t(C_n^2) = nF_{-}\{n\}, \quad n \geqslant 5, \tag{5}$$

where $F_{-}\{n\}$ is the n'th Fibonacci Number. Similar results can also be found in [6].

The next result is due to Boesch and Bogdanowicz [7] who derived in 1987 a formula for the prism on $2n$ vertices, R_n, which is formed from two disjoint cycles C_n with vertex set $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and C'_n with vertex set $V(C'_n) = \{v'_1, v'_2, \cdots, v'_n\}$ by adding all edges of the form $v_i v'_i$. That is

$$t(R_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2] \quad \text{for} \quad n \geqslant 3. \tag{6}$$

In 2007, Bleller and Saccoman [8] derived a formula for a threshold graph on n vertices, T_n, which is formed from that for all pairs of vertices u and v in T_n, $N(u)-v\subset N(v)-u$ whenever $d(u)\leqslant d(v)$. That is

$$t(T_n)=\prod_{i=1}^{k}d_i^{n_i}\prod_{j=k+2}^{s-1}(d_j+1)^{n_j}(d_{k+1}+l)^{n_{k+1}-1}(d_s+1)^{n_s-1}, \tag{7}$$

where $\overline{d}(T_n)=(d_1^{(n_1)},d_2^{(n_2)},\cdots,d_s^{(n_s)})$ is the degree sequence of T_n, $d_i<d_{i+1}$ for $i=1,2,\cdots,s-1$, $k=\lfloor\frac{s-1}{2}\rfloor$ and $s\equiv l(\bmod\ 2)$.

In 2008, Bogdanowicz [9] also derived a formula for an n-fan on $n+1$ vertices, F_{n+1}, which is formed from a n-path P_n by adding an additional vertex adjacent to every vertex of P_n. That is

$$t(F_{n+1})=\frac{2}{5-3\sqrt{5}}[(\frac{3-\sqrt{5}}{2})^{n+1}-(\frac{3+\sqrt{5}}{2})^{n-1}]\quad\text{for}\quad n\geqslant 2. \tag{8}$$

In this paper it is proved that: Let $K_{k,\overline{n}}$ be a complete k-partite graph of order n where the vector $\overline{n}=\{n_1,n_2,\cdots,n_k\}$ and $n=n_1+n_2+\cdots+n_k$ for $1\leqslant k\leqslant n$, then the number of spanning trees in $K_{k,\overline{n}}$ is

$$t(K_{k,\overline{n}})=n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}. \tag{9}$$

Moreover, let $K_{k,\overline{n}}^{\mathscr{F}}$ be a generalized complete k-partite graph of order n spanned by the fan set $\mathscr{F}=\{F_{n_1},F_{n_2},\cdots,F_{n_k}\}$ where $\overline{n}=\{n_1,n_2,\cdots,n_k\}$ and $n=n_1+n_2+\cdots+n_k$ for $1\leqslant k\leqslant n$, then the number of spanning trees in $K_{k,\overline{n}}^{\mathscr{F}}$ is

$$t(K_{k,\overline{n}}^{\mathscr{F}})=n^{2k-2}\prod_{i=1}^{k}\frac{\alpha_i^{n_i-1}-\beta_i^{n_i-1}}{\alpha_i-\beta_i} \tag{10}$$

where $\alpha_i=(d_i+\sqrt{d_i^2-4})/2$ and $\beta_i=(d_i-\sqrt{d_i^2-4})/2$,$d_i=n-n_i+3$.

In particular, from (9) we can obtain $K_{1,\overline{n}}=K_n^c$ with $t(K_n^c)=0$ and $K_{n,\overline{n}}=K_n$ with $t(K_n)=n^{n-2}$ which is just the Cayley's formula (2). From (10) we can also obtain $K_{1,\overline{n}}^{\mathscr{F}}=F_n$ with $t(K_{1,\overline{n}}^{\mathscr{F}})=(\alpha^{n-1}-\beta^{n-1})/\sqrt{5}$ where $\alpha=\frac{1}{2}(3+\sqrt{5})$ and $\beta=\frac{1}{2}(3-\sqrt{5})$ which is the formula (8), too.

§2. Some Lemmas

In order to calculate the number of spanning trees of G, we first denote by $A(G)$, or $A=(a_{ij})_{n\times n}$, the *adjacency matrix* of G, which has the rows and columns corresponding to the vertices, and entries $a_{ij}=1$ if there is an edge between vertices v_i and v_j in $V(G)$, $a_{ij}=0$ otherwise.

In addition, let $D(G)$ represent the diagonal matrix of the degrees of the vertices of G. We denote by $H(G)$, or $H=(h_{ij})_{n\times n}$, the *Laplacian matrix* (also known as the nodal admittance matrix) $D(G)-A(G)$ of G. From $H(G)=D(G)-A(G)$, we can see that $h_{ii}=d(v_i)$ and $h_{ij}=-a_{ij}$ if $i\neq j$.

A well-known result states the relationship between the number of spanning trees of a graph and the eigenvalues of its nodal admittance matrix:

Lemma 2.1([10]) *The value of $t(G)$, the number of spanning trees of a graph G, is related to the eigenvalues $\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)$ of its nodal admittance matrix $H = H(G)$ as follows:*

$$t(G) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i(G), \qquad 0 = \lambda_1(G) \leqslant \lambda_2(G) \leqslant \cdots \leqslant \lambda_n(G). \tag{11}$$

A classic result of Kirchhoff can be used to determine the number of spanning trees for G. Next, we state the well-known theorem of Kirchhoff:

Lemma 2.2(Kirchhoff's Matrix-Tree Theorem, [11]) *All cofactors of H are equal and their common value is the number of spanning trees.*

Lemma 2.3 *Let $b > 2$ be a constant, then the determinant of order m*

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & b & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & b & -1 \\ 0 & \cdots & 0 & -1 & b-1 \end{vmatrix}_{m\times m} = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \tag{12}$$

where $\alpha = (b + \sqrt{b^2-4})/2$, $\beta = (b - \sqrt{b^2-4})/2$.

Proof Let a_m stand for the determinant in (**??**) as above, by expending the determinant according to the first column we then have

$$a_m = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & b & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & b & -1 \\ 0 & \cdots & 0 & -1 & b-1 \end{vmatrix}_{m\times m} = c_{m-1} + a_{m-1}, \tag{13}$$

where

$$c_m = \begin{vmatrix} b & -1 & 0 & \cdots & 0 \\ -1 & b & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & b & -1 \\ 0 & \cdots & 0 & -1 & b-1 \end{vmatrix}_{m\times m} = bc_{m-1} - c_{m-2} \tag{14}$$

in which $c_0 = 1$, $c_1 = b - 1$ and $c_2 = b^2 - b - 1$.

Constructing a function as follows:

$$f(x) = \sum_{m\geqslant 0} c_m x^m, \tag{15}$$

then

$$\begin{aligned} f(x) &= 1 + (b-1)x + \sum_{m\geqslant 2} c_m x^m \\ &= 1 + (b-1)x + \sum_{m\geqslant 2} (bc_{m-1} - c_{m-2})x^m \\ &= 1 + (b-1)x + bx(f(x) - 1) - x^2 f(x), \end{aligned}$$

i.e.

$$f(x) = \frac{1-x}{x^2 - bx + 1} = \sum_{m\geqslant 0} \frac{\alpha^{m+1} + \beta^m}{1+\alpha} x^m \tag{16}$$

where $\alpha = (b + \sqrt{b^2-4})/2$ and $\beta = (b - \sqrt{b^2-4})/2$ which is just the two solutions of the equation $x^2 - bx + 1 = 0$.

Thus, from (11.4) and (11.5) we have

$$c_m = \frac{\alpha^{m+1} + \beta^m}{1+\alpha}. \tag{17}$$

Since

$$a_3 = \begin{vmatrix} 1 & 1 & 1 \\ -1 & b & -1 \\ 0 & -1 & b-1 \end{vmatrix} = b^2 - 1 = c_2 + c_1 + c_0, \tag{18}$$

from (11.2),(11.6) and (11.7) we can obtain that

$$a_m = \sum_{k=0}^{m-1} c_k = \sum_{k=0}^{m-1} \frac{\alpha^{k+1} + \beta^k}{1+\alpha} = \frac{\alpha^m - \beta^m}{\alpha - \beta}.$$

§3 Complete Multipartite Graphs

For a complete k-partite graph $K_{k,\overline{n}}$ of order n where the vector $\overline{n} = \{n_1, n_2, \cdots, n_k\}$ and $n = n_1 + n_2 + \cdots + n_k$ for $1 \leqslant k \leqslant n$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Let $V(K_{k,\overline{n}}) = V_1 \cup V_2 \cup \cdots \cup V_k$ be the k partitions of the graph $K_{k,\overline{n}}$ such that $|V_i| = n_i$ for $i = 1, 2, \cdots, k$. It is clear that for any vertex $v_i \in V_i$, $d(v_i) = n - n_i$ for $i = 1, 2, \cdots, k$. So the degree sequence of vertices in $K_{k,\overline{n}}$

$$\overline{d}(K_{k,\overline{n}}) = ((n-n_1)^{n_1}, (n-n_2)^{n_2}, \cdots, (n-n_k)^{n_k}). \tag{19}$$

Now, let E_i be an identity matrix of order i, $I_{i\times j} = (1)_{i\times j}$ a total module matrix of order $i \times j$ and $O_{i\times j} = (0)_{i\times j}$ a total zero matrix of order $i \times j$. Then the diagonal block matrix of the degrees of the vertices of $K_{k,\overline{n}}$ corresponding to (12) is

$$D(K_{k,\overline{n}}) = (D_{ij})_{k\times k}, \tag{20}$$

where $D_{ii} = (n - n_i)E_{n_i}$ and $D_{ij} = O_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$. The adjacency matrix of $K_{k,\overline{n}}$ corresponding to (**??**) is

$$A(K_{k,\overline{n}}) = (A_{ij})_{k \times k}, \tag{21}$$

where $A_{ii} = O_{n_i}$ and $A_{ij} = I_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

Thus, we have from (13) and (14) the Laplacian block matrix (or the nodal admittance matrix) of $K_{k,\overline{n}}$

$$H(K_{k,\overline{n}}) = D(K_{k,\overline{n}}) - A(K_{k,\overline{n}}) = (H_{ij})_{k \times k}, \tag{22}$$

where $H_{ii} = (n - n_i)E_{n_i}$ and $H_{ij} = -I_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$. □

Theorem 3.1 *Let $K_{k,\overline{n}}$ be a complete k-partite graph of order n where the vector $\overline{n} = \{n_1, n_2, \cdots, n_k\}$ and $n = n_1 + n_2 + \cdots + n_k$ for $1 \leqslant k \leqslant n$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Then the number of spanning trees of $K_{k,\overline{n}}$ is*

$$t(K_{k,\overline{n}}) = n^{k-2} \prod_{i=1}^{k} (n - n_i)^{n_i - 1}. \tag{23}$$

Proof According to Lemma 2.1, we need to determine all eigenvalues of nodal admittance matrix $H(K_{k,\overline{n}})$ of $K_{k,\overline{n}}$. From (15) we can get easily the characteristic polynomial of $H(K_{k,\overline{n}})$ as follows:

$$|H(K_{k,\overline{n}}) - \lambda E| = \begin{vmatrix} (n-n_1-\lambda)E_{n_1} & -I_{n_1 \times n_2} & \cdots & -I_{n_1 \times n_k} \\ -I_{n_2 \times n_1} & (n-n_2-\lambda)E_{n_2} & \cdots & -I_{n_2 \times n_k} \\ \cdots & \cdots & \cdots & \cdots \\ -I_{n_k \times n_1} & -I_{n_k \times n_2} & \cdots & (n-n_k-\lambda)E_{n_k} \end{vmatrix}_{n \times n}. \tag{24}$$

Since the summations of entries in every column in $|H(K_{k,\overline{n}}) - \lambda E|$ all are $-\lambda$, by adding the entries of all rows other than the first row to the first row in $|H(K_{k,\overline{n}}) - \lambda E|$, all entries of the first row then become $-\lambda$. Thus, the determinant becomes

$$|H(K_{k,\overline{n}}) - \lambda E| = -\lambda \begin{vmatrix} H^+_{n_1} & * & \cdots & * \\ -I_{n_2 \times n_1} & (n-n_2-\lambda)E_{n_2} & \cdots & -I_{n_2 \times n_k} \\ \cdots & \cdots & \cdots & \cdots \\ -I_{n_k \times n_1} & -I_{n_k \times n_2} & \cdots & (n-n_k-\lambda)E_{n_k} \end{vmatrix}_{n \times n}, \tag{25}$$

where

$$H^+_{n_1} = \begin{pmatrix} 1 & I_{1 \times (n_1 - 1)} \\ O_{(n_1-1) \times 1} & (n - n_1 - \lambda)E_{(n_1-1)} \end{pmatrix}_{n_1 \times n_1} \tag{26}$$

and the stars "*" in (18) stand for the matrices with all entries in the first row being 1. By adding the entries in the first row to the rows from row n_1+1 to row n in the determinant (18), it then becomes

$$|H(K_{k,\overline{n}})-\lambda E| =$$

$$-\lambda\begin{vmatrix} H^+_{n_1} & * & \cdots & * \\ O_{n_2\times n_1} & (n-n_2-\lambda)E_{n_2}+I_{n_2} & \cdots & O_{n_2\times n_k} \\ \cdots & \cdots & \cdots & \cdots \\ O_{n_k\times n_1} & O_{n_k\times n_2} & \cdots & (n-n_k-\lambda)E_{n_k}+I_{n_k} \end{vmatrix}_{n\times n},$$

i.e.

$$|H(K_{k,\overline{n}})-\lambda E| = -\lambda|H^+_{n_1}|\prod_{1<i\leqslant k}|(n-n_i-\lambda)E_{n_i}+I_{n_i}|. \tag{27}$$

From (19) we have

$$|H^+_{n_1}| = \begin{vmatrix} 1 & I_{1\times n_1} \\ O_{n_1\times 1} & (n-n_1-\lambda)E_{n_1} \end{vmatrix} = (n-n_1-\lambda)^{n_1-1}. \tag{28}$$

For $1<i\leqslant k$ we have

$$|(n-n_i-\lambda)E_{n_i}+I_{n_i}| = \begin{vmatrix} n-\lambda & (n-\lambda)I_{1\times(n_i-1)} \\ I_{(n_i-1)\times 1} & (n-n_i-\lambda)E_{n_i-1}+I_{n_i-1} \end{vmatrix}$$

i.e.

$$\begin{aligned}|(n-n_i-\lambda)E_{n_i}+I_{n_i}| &= (n-\lambda)\begin{vmatrix} 1 & I_{1\times(n_i-1)} \\ O_{(n_i-1)\times 1} & (n-n_i-\lambda)E_{n_i-1} \end{vmatrix} \\ &= (n-\lambda)(n-n_i-\lambda)^{n_i-1}.\end{aligned} \tag{29}$$

By substituting (21),(22) to (20) we have

$$|H(K_{k,\overline{n}})-\lambda E| = -\lambda(n-\lambda)^{k-1}\prod_{i=1}^{k}(n-n_i-\lambda)^{n_i-1}. \tag{30}$$

So, from (23) we derive all eigenvalues of $H(K_{k,\overline{n}})$ as follows: $\lambda_1=0$ and

$$\begin{cases} (n_i-1)\text{-multiple roots } \ \lambda_{i+1}(K_{k,\overline{n}}) = n-n_i, \quad i=1,2,\cdots,k; \\ \\ (k-1)\text{-multiple root } \ \lambda_n(K_{k,\overline{n}}) = n. \end{cases} \tag{31}$$

By substituting (24) to (11) we have

$$t(K_{k,\overline{n}}) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i(K_{k,\overline{n}}) = n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}. \tag{32}$$

This is just the theorem. □

Corollary 3.2 (Cayley's formula) *The total number of spanning trees of complete graph K_n is*

$$t(K_n) = n^{n-2}. \tag{33}$$

Proof Let $k = n$, then $n_1 = n_2 = \cdots = n_n = 1$ and $K_{n,\overline{n}} = K_n$. By substituting it to (25) we then have

$$t(K_n) = t(K_{n,\overline{n}}) = n^{n-2}\prod_{i=1}^{n}(n-1)^0 = n^{n-2}. \quad \square$$

§3. Generalized Complete Multipartite Graphs

For a generalized complete k-partite graph $K_{k,\overline{n}}^{\mathscr{F}}$ of order n spanned by the fan set $\mathscr{F} = \{F_{n_1}, F_{n_2}, \cdots, F_{n_k}\}$ where $n = n_1 + n_2 + \cdots + n_k$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$ for $1 \leqslant k \leqslant n$. It is clear that the degree sequence of vertices of the fan F_{n_i} in $K_{k,\overline{n}}^{\mathscr{F}}$

$$\overline{d}(F_{n_i}) = (n - n_i + 2, (n - n_i + 3)^{n_i - 3}, n - n_i + 2, n - 1) \tag{34}$$

for $i = 1, 2, \cdots, k$. So the degree sequence of vertices of $K_{k,\overline{n}}^{\mathscr{F}}$ is

$$\overline{d}(K_{k,\overline{n}}^{\mathscr{F}}) = (\overline{d}(F_{n_1}), \overline{d}(F_{n_2}), \cdots, \overline{d}(F_{n_k})). \tag{35}$$

Now, the diagonal block matrix of the degrees of the vertices of $K_{k,\overline{n}}^{\mathscr{F}}$ corresponding to (28) is

$$D(K_{k,\overline{n}}^{\mathscr{F}}) = (D_{ij})_{k\times k}, \tag{36}$$

where from (27) we have

$$D_{ii} = \mathrm{diag}\{n - n_i + 2, n - n_i + 3, \cdots, n - n_i + 3, n - n_i + 2, n - 1\} \tag{37}$$

and $D_{ij} = O_{n_i\times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

The adjacency matrix of $K_{k,\overline{n}}^{\mathscr{F}}$ corresponding to (28) is

$$A(K_{k,\overline{n}}^{\mathscr{F}}) = (A_{ij})_{k\times k}, \tag{38}$$

where

$$A_{ii} = \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & \ddots & & 1 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n_i\times n_i} \tag{39}$$

and $A_{ij} = I_{n_i\times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

Thus, we have from (29) $\sim$ (32) the Laplacian block matrix (or the nodal admittance matrix) of $K_{k,\overline{n}}^{\mathscr{F}}$

$$H(K_{k,\overline{n}}^{\mathscr{F}}) = D(K_{k,\overline{n}}^{\mathscr{F}}) - A(K_{k,\overline{n}}^{\mathscr{F}}) = (H_{ij})_{k\times k}, \tag{40}$$

where

$$H_{ii} = D_{ii} - A_{ii} =$$

$$\begin{pmatrix} d_i-1 & -1 & & & & -1 \\ -1 & d_i & -1 & & & -1 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -1 & d_i & -1 & \\ & & & -1 & d_i-1 & -1 \\ -1 & -1 & \cdots & & -1 & n-1 \end{pmatrix}_{n_i\times n_i} \tag{41}$$

with $d_i = n - n_i + 3$ and $H_{ij} = -I_{n_i\times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

Theorem 4.1 *Let $K_{k,\overline{n}}^{\mathscr{F}}$ be a generalized complete k-partite graph of order n spanned by the fan set $\mathscr{F} = \{F_{n_1}, F_{n_2}, \cdots, F_{n_k}\}$ where $\overline{n} = \{n_1, n_2, \cdots, n_k\}$ and $n = n_1 + n_2 + \cdots + n_k$ for $1 \leqslant k \leqslant n$, then the number of spanning trees in $K_{k,\overline{n}}^{\mathscr{F}}$ is*

$$t(K_{k,\overline{n}}^{\mathscr{F}}) = n^{2k-2}\prod_{i=1}^{k}\frac{\alpha_i^{n_i-1} - \beta_i^{n_i-1}}{\alpha_i - \beta_i} \tag{42}$$

where $\alpha_i = (d_i + \sqrt{d_i^2 - 4})/2$ and $\beta_i = (d_i - \sqrt{d_i^2 - 4})/2$,$d_i = n - n_i + 3$.

Proof According to the Kirchhoff's Matrix-Tree theorem, all cofactors of $H(K_{k,\overline{n}}^{\mathscr{F}})$ are equal to the number of spanning trees $t(K_{k,\overline{n}}^{\mathscr{F}})$. Let $H^*(K_{k,\overline{n}}^{\mathscr{F}})$ be the cofactor of $H(K_{k,\overline{n}}^{\mathscr{F}})$ corresponding to the entry h_{nn} of $H(K_{k,\overline{n}}^{\mathscr{F}})$, then

$$t(K_{k,\overline{n}}^{\mathscr{F}}) = |H^*(K_{k,\overline{n}}^{\mathscr{F}})| =$$

$$\begin{vmatrix} H_{11} & \cdots & -I_{n_1\times n_i} & \cdots & -I_{n_1\times(n_k-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -I_{n_i\times n_1} & \cdots & H_{ii} & \cdots & -I_{n_i\times(n_k-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -I_{(n_k-1)\times n_1} & \cdots & -I_{(n_k-1)\times n_i} & \cdots & H_{kk}^* \end{vmatrix} \tag{43}$$

where

$$H^*_{kk} = \begin{pmatrix} d_k-1 & -1 & & & \\ -1 & d_k & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & d_k & -1 \\ & & & -1 & d_k-1 \end{pmatrix}_{(n_k-1)\times(n_k-1)}. \quad (44)$$

Since the summations of entries in every column in $|H^*(K^{\mathscr{F}}_{k,\overline{n}})|$ all are 1 by adding the entries of all rows other than the first row to the first row in $|H^*(K^{\mathscr{F}}_{k,\overline{n}})|$ the entries of the first row become 1. By adding the entries in the first row to the rows from row n_1+1 to row $n-1$ in the last determinant, it becomes from (36) and (37)

$$|H^*(K^{\mathscr{F}}_{k,\overline{n}})| = \begin{vmatrix} H^*_{n_1} & \cdots & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O_{n_i\times n_1} & \cdots & H_{ii}+I_{n_i} & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O_{(n_k-1)\times n_1} & \cdots & O_{(n_k-1)\times n_i} & \cdots & H^*_{n_k-1} \end{vmatrix},$$

i.e.

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = |H^*(K^{\mathscr{F}}_{k,\overline{n}})| = |H^*_{n_1}| \cdot \prod_{1<i<k} |H_{ii}+I_{n_i}| \cdot |H^*_{n_k-1}|, \quad (45)$$

where

$$|H^*_{n_1}| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & d_1 & -1 & & & -1 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -1 & d_1 & -1 & -1 \\ & & & -1 & d_1-1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & n-1 \end{vmatrix}_{n_1\times n_1},$$

by adding the entries in the first row to the correspondence entries of the other row in $|H^*_{n_1}|$ we then have

$$|H^*_{n_1}| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & d_1+1 & 0 & 1 & \vdots & 0 \\ 1 & \ddots & \ddots & \ddots & 1 & \vdots \\ \vdots & 1 & 0 & d_1+1 & 0 & 0 \\ 1 & \cdots & 1 & 0 & d_1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & n \end{vmatrix}_{n_1\times n_1},$$

i.e.

$$|H^*_{n_1}| = n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_1+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_1+1 & 0 \\ 1 & \cdots & 1 & 0 & d_1 \end{vmatrix}_{(n_1-1)\times(n_1-1)} ; \qquad (46)$$

It is clear that

$$|H_{ii} + I_{n_i}| = \begin{vmatrix} d_i & 0 & 1 & \cdots & 1 & 0 \\ 0 & d_i+1 & 0 & 1 & \vdots & \\ 1 & \ddots & \ddots & \ddots & 1 & \vdots \\ \vdots & 1 & 0 & d_i+1 & 0 & \\ 1 & \cdots & 1 & 0 & d_i & 0 \\ 0 & & \cdots & & 0 & n \end{vmatrix}_{n_i\times n_i} ,$$

i.e.

$$|H_{ii} + I_{n_i}| = n \begin{vmatrix} d_i & 0 & 1 & \cdots & 1 \\ 0 & d_i+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_i+1 & 0 \\ 1 & \cdots & 1 & 0 & d_i \end{vmatrix}_{(n_i-1)\times(n_i-1)} .$$

Since the summations of all entries in every column in the last determinant, we have

$$|H_{ii} + I_{n_i}| = n^2 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_i+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_i+1 & 0 \\ 1 & \cdots & 1 & 0 & d_i \end{vmatrix}_{(n_i-1)\times(n_i-1)} ; \qquad (47)$$

and

$$|H^*_{n_k-1}| = \begin{vmatrix} d_k & 0 & 1 & \cdots & 1 \\ 0 & d_k+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_k+1 & 0 \\ 1 & \cdots & 1 & 0 & d_k \end{vmatrix}_{(n_k-1)\times(n_k-1)} ,$$

Similarly, we have

$$|H^*_{n_k-1}| = n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_k+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_k+1 & 0 \\ 1 & \cdots & 1 & 0 & d_k \end{vmatrix}_{(n_k-1)\times(n_k-1)}. \tag{48}$$

By substituting (39),(40) and (41) to (38), it becomes

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2} \prod_{1\leqslant i\leqslant k} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_i+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_i+1 & 0 \\ 1 & \cdots & 1 & 0 & d_i \end{vmatrix}_{(n_i-1)\times(n_i-1)},$$

or

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2} \prod_{1\leqslant i\leqslant k} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & d_i & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & d_i & -1 \\ 0 & \cdots & 0 & -1 & d_i-1 \end{vmatrix}_{(n_i-1)\times(n_i-1)}. \tag{49}$$

Thus, from (11.1) and (42) we have

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2} \prod_{1\leqslant i\leqslant k} \frac{\alpha_i^{n_i-1} - \beta_i^{n_i-1}}{\alpha_i - \beta_i} \tag{50}$$

where $\alpha_i = (d_i + \sqrt{d_i^2 - 4})/2$, $\beta_i = (d_i - \sqrt{d_i^2 - 4})/2$ and $d_i = n - n_i + 3$.

This is just the theorem. □

Corollary 4.2 *The total number of spanning trees of a fan graph F_n is*

$$t(F_n) = \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1}) \tag{51}$$

where $\alpha = (3+\sqrt{5})/2$ *and* $\beta = (3-\sqrt{5})/2$.

Proof Let $k = 1$, then $n_1 = n$ and $K^{\mathscr{F}}_{1,\overline{n}} = F_n$. Substituting this fact into (35), this result is followed. □

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