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# On the Product of the Square-free Divisor of a Natural Number

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**Abstract** In this paper we study the product of the square-free divisor of a natural number $P_{sd}(n) = \prod_{\substack{d|n \\ \mu(d)\neq 0}} d$. According to the Dirichlet divisor problem, we turn to study the asymptotic formula of $\sum_{n\leq x} \log P_{sd}(n)$. This article uses the hyperbolic summation and the convolution method to obtain a better error term.

**Keywords** Square-free number, Dirichlet divisor problem, hyperbolic summation, convolution.

## §1. Introduction and main results

F.Smarandache introduced the function $P_d(n) := \prod_{d|n} d$ in Problem 25[1]. Now we define a similar function $P_{sd}(n)$, which denotes the product of all square-free divisors of $n$, i.e.,

$$P_{sd}(n) := \prod_{\substack{d|n \\ \mu(d)\neq 0}} d.$$

In the present paper, we shall prove the following Theorem.

**Theorem.** We have the asymptotic formula

$$\sum_{n\leq x} \log P_{sd}(n) = A_1 x \log^2 x + A_2 x \log x + A_3 x + O(x^{\frac{1}{2}} \mathtt{exp}(-D(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}),$$

where $A_1$, $A_2$, $A_3$ are constants, $D > 0$ is an absolute constant.

**Notations.** $[x] = max_{k\in Z}\{k \leq x\}.\psi(t) = t - [t] - \frac{1}{2}, \psi_1(t) = \int_0^t \psi(u)du.$ $\mu(n)$ is the $Mobius$ function. $\varepsilon$ denotes a fixed small positive constant which may be different at each occurence. $\gamma$ is the Euler constant. $B_1$, $B_2$, $B_3$, $C_1$, $C_2$ , $D_1$, $D_2$, $D_3$, $D_4$ are constants.

## §2. Some preliminary lemmas

We need the following results:

**Lemma 1.** Let $f(n)$ be an arithmetical function for which

$$\sum_{n\leq x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a),$$

$$\sum_{n\le x}|f(n)| = O(x^{a_1}\log^r x),$$

where $a_1 \ge a_2 \ge \cdots \ge a_l > \frac{1}{k} > a \ge 0$, $r \ge 0$, $P_1(t), \cdots, P_l(t)$ are polynomials in $t$ of degrees not exceeding $r$ , and $k \ge 1$ is a fixed integer. If

$$h(n) = \sum_{d^k|n}\mu_i(d)f(n/d^k), \quad i \ge 1$$

then

$$\sum_{n\le x}h(n) = \sum_{j=1}^{l}x^{a_j}R_j(\log x) + \delta(x),$$

where $R_1(t), \cdots, R_l(t)$ are polynomials in $t$ of degrees not exceeding $r$. and for some $D > 0$

$$\delta(x) \ll x^{\frac{1}{k}}\exp\left(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right).$$

Proof. See Theorem 14.2 of A. Ivić[3] when $l = 1$ and the similar proof is used when $l > 1$ .

**Lemma 2.** Suppose that $f(u) \in C^3[u_1, u_2]$, then

$$\sum_{u_1<n\le u_2}f(n) = \int_{u_1}^{u_2}f(u)du - \psi(u)f(u)\Big|_{u_1}^{u_2} + \psi_1(u)f'(u)\Big|_{u_1}^{u_2} - \int_{u_1}^{u_2}\psi_1(u)f''(u)du.$$

**Lemma 3.** We have

$$\sum_{1\le m\le y}\frac{1}{m} = \log y + \gamma - \frac{\psi(y)}{y} + O\left(\frac{1}{y^2}\right), \ y \ge 1.$$

**Lemma 4.** We have

$$\sum_{1\le m\le y}\log m = y\log y - y - \psi(y)\log y + \frac{\psi_1(y)}{y} + D_1 + O\left(\frac{1}{y^2}\right), \ y \ge 1.$$

**Lemma 5.** We have

$$\sum_{1\le m\le y}\frac{\log m}{m^2} = -\frac{\log y}{y} - \frac{1}{y} + D_3 + O\left(\frac{\log y}{y^2}\right), \ y \ge 1.$$

**Lemma 6.** We have

$$\sum_{1\le m\le y}\frac{\log^2 m}{m^2} = -\frac{\log^2 y}{y} - \frac{2\log y}{y} - \frac{2}{y} + D_4 + O\left(\frac{\log^2 y}{y^2}\right), \ y \ge 1.$$

**Lemma 7.** We have

$$\sum_{n\le y}d(n) = y\log y + (2\gamma - 1)y + O(y^{\frac{1}{3}}).$$

**Lemma 8.** We have

$$\sum_{n\le y}d(n)n^{-\frac{1}{2}} = 2y^{\frac{1}{2}}\log y + (4\gamma - 4)y^{\frac{1}{2}} - 2\gamma + 3 + O(y^{-\frac{1}{6}}).$$

**Lemma 9.** We have

$$\sum_{n\leq y} d(n)n^{-\frac{1}{2}}\log n = 2y^{\frac{1}{2}}\log^2 y + (4\gamma-8)y^{\frac{1}{2}}\log y + (16-8\gamma)y^{\frac{1}{2}} + 8\gamma - 16 + O(y^{-\frac{1}{6}}\log y).$$

Lemma 2 is the Euler-Maclaurin summation formula (see [2]). Lemma 3, 4, 5, 6 follow from Lemma 2 directly. Lemma 7 is a classical result about the Dirichlet divisor problem. Lemma 8, 9 can be easily obtained by Lemma 2 and Lemma 7.

## §3. Proof of the theorem

It is easily seen that

$$\log P_{sd}(n) = \sum_{n=dl,\mu(d)\neq 0} \log d,$$

which implies that ($\sigma > 1$)

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\log P_{sd}(n)}{n^s} &= \zeta(s)\sum_{d=1}^{\infty}\frac{|\mu(d)|\log d}{d^s} = \zeta(s)\left(-\sum_{d=1}^{\infty}\frac{|\mu(d)|}{d^s}\right)' \\ &= -\zeta(s)\left(\frac{\zeta(s)}{\zeta(2s)}\right)' = -\zeta(s)\zeta'(s)\frac{1}{\zeta(2s)} + 2\zeta^2(s)\zeta'(2s)\frac{1}{\zeta^2(2s)} \\ &= \sum_{n=1}^{\infty} h_1(n)n^{-s} + 2\sum_{n=1}^{\infty} h_2(n)n^{-s}.\end{aligned}$$

where

$$h_1(n) = \sum_{d^2|n}\mu(d)f_1(n/d^2), f_1(n) = \sum_{m|n}\log m;$$

$$h_2(n) = \sum_{d^2|n}\mu_2(d)f_2(n/d^2), \quad f_2(n) = \sum_{n=m^2k} d(k)\log m, \quad \mu_2(d) = \sum_{n=dk}\mu(d)\mu(k).$$

Our Theorem follows from the following Proposition 1 and Proposition 2.

**Proposition 1.** We have

$$\sum_{n\leq x} h_1(n) = B_1 x\log^2 x + B_2 x\log x + B_3 x + O(x^{\frac{1}{2}}\mathtt{exp}(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}).$$

**Proposition 2.** We have

$$\sum_{n\leq x} h_2(n) = C_1 x\log x + C_2 x + O(x^{\frac{1}{2}}\mathtt{exp}(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}).$$

We only proof Proposition 2. The proof of Proposition 1 is similar and easier. We have

$$\begin{aligned}&\sum_{n\leq x} f_2(n) = \sum_{n\leq x}\sum_{m^2k=n} d(k)\log m = \sum_{m^2n\leq x} d(n)\log m \\ &= \sum_{m\leq x^{\frac{1}{3}}}\log m\sum_{n\leq\frac{x}{m^2}} d(n) + \sum_{n\leq x^{\frac{1}{3}}} d(n)\sum_{m\leq(\frac{x}{n})^{\frac{1}{2}}}\log m - \sum_{m\leq x^{\frac{1}{3}}}\log m\sum_{n\leq x^{\frac{1}{3}}} d(n) \\ &= S_1 + S_2 - S_3\end{aligned}$$

By Lemma 7, 5, 6

$$
\begin{aligned}
S_1 &= \sum_{m\le x^{\frac{1}{3}}}\left(\frac{x}{m^2}\log\frac{x}{m^2}+(2\gamma-1)\frac{x}{m^2}+O\left(\left(\frac{x}{m^2}\right)^{\frac{1}{3}}\right)\right)\log m\\
&=(x\log x+(2\gamma-1)x)\sum_{m\le x^{\frac{1}{3}}}\frac{\log m}{m^2}-2x\sum_{m\le x^{\frac{1}{3}}}\frac{\log^2 m}{m^2}+O\left(x^{\frac{1}{3}}\sum_{m\le x^{\frac{1}{3}}}\frac{\log m}{m^{\frac{2}{3}}}\right)\\
&=(x\log x+(2\gamma-1)x)\left(-\frac{1}{3}x^{-\frac{1}{3}}\log x-x^{-\frac{1}{3}}+D_3+O\left(x^{-\frac{2}{3}}\log x\right)\right)\\
&\quad -2x\left(-\frac{1}{9}x^{-\frac{1}{3}}\log^2 x-\frac{2}{3}x^{-\frac{1}{3}}\log x-2x^{-\frac{1}{3}}+D_4+O\left(x^{-\frac{2}{3}}\log x\right)\right)+O\left(x^{\frac{4}{9}+\varepsilon}\right)
\end{aligned}
$$

By Lemma 4, 8, 9

$$
\begin{aligned}
S_2 &= \sum_{n\le x^{\frac{1}{3}}}d(n)\left(\frac{1}{2}\left(\frac{x}{n}\right)^{\frac{1}{2}}\log\frac{x}{n}-\left(\frac{x}{n}\right)^{\frac{1}{2}}+D_1+O\left(\log\frac{x}{n}\right)\right)\\
&=(\frac{1}{2}x^{\frac{1}{2}}\log x-x^{\frac{1}{2}})\sum_{n\le x^{\frac{1}{3}}}d(n)n^{-\frac{1}{2}}-\frac{1}{2}x^{\frac{1}{2}}\sum_{n\le x^{\frac{1}{3}}}d(n)\log n\, n^{-\frac{1}{2}}\\
&\quad +D_1\sum_{n\le x^{\frac{1}{3}}}d(n)+O\left(\log x\sum_{n\le x^{\frac{1}{3}}}d(n)\right)\\
&=\left(\frac{1}{2}x^{\frac{1}{2}}\log x-x^{\frac{1}{2}}\right)\left(\frac{2}{3}x^{\frac{1}{6}}\log x+(4\gamma-4)x^{\frac{1}{6}}-2\gamma+3+O\left(x^{-\frac{1}{18}}\right)\right)\\
&\quad -\frac{1}{2}x^{\frac{1}{2}}\left(\frac{2}{9}x^{\frac{1}{6}}\log^2 x+\frac{1}{3}(4\gamma-8)x^{\frac{1}{6}}\log x+(16-8\gamma)x^{\frac{1}{6}}+8\gamma-16+O\left(x^{-\frac{1}{18}}\log x\right)\right)\\
&\quad +D_1\sum_{n\le x^{\frac{1}{3}}}d(n)+O\left(x^{\frac{1}{3}+\varepsilon}\right)
\end{aligned}
$$

By Lemma 4 and Lemma 7,

$$
\begin{aligned}
S_3 &= \sum_{n\le x^{\frac{1}{3}}}d(n)\left(\frac{1}{3}x^{\frac{1}{3}}\log x-x^{\frac{1}{3}}+D_1+O\left(\log x\right)\right)\\
&=\frac{1}{3}x^{\frac{1}{3}}\log x\sum_{n\le x^{\frac{1}{3}}}d(n)-x^{\frac{1}{3}}\sum_{n\le x^{\frac{1}{3}}}d(n)+D_1\sum_{n\le x^{\frac{1}{3}}}d(n)\\
&\quad +O\left(\log x\sum_{n\le x^{\frac{1}{3}}}d(n)\right)\\
&=\left(\frac{1}{3}x^{\frac{1}{3}}\log x-x^{\frac{1}{3}}\right)\left(\frac{1}{3}x^{\frac{1}{3}}\log x+(2\gamma-1)x^{\frac{1}{3}}+O\left(x^{\frac{1}{9}}\right)\right)\\
&\quad +O\left(x^{\frac{1}{3}+\varepsilon}\right)+D_1\sum_{n\le x^{\frac{1}{3}}}d(n)
\end{aligned}
$$

Combining the above estimates we get

$$\sum_{n\leq x} f_2(n) = D_3 x \log x + ((2\gamma-1)D_3 - 2D_4)\, x + \frac{3-2\gamma}{2} x^{\frac{1}{2}} \log x + (5-2\gamma) x^{\frac{1}{2}} + O\left(x^{\frac{4}{9}+\varepsilon}\right),$$

which gives Proposition 2 immediately by using Lemma 1 .

# References

[1] Smarandache F, Only problems, not Solutions, Chicago, Xiquan Publ, House, 1993.

[2] Chengdong Pan and Chengbiao Pan, Foundation of Analytic Number Theory, Science Press, Beijing, 1997.

[3] A. Ivić, the Riemann zeta-function, Wiley-Interscience, New York, 1985.