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An equation involving the square sum of natural numbers and Smarandache primitive function

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Abstract For any positive integer n, let $S_p(n)$ denotes the Smarandache primitive function. The main purpose of this paper is using the elementary methods to study the number of the solutions of the equation $S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right)$, and give all positive integer solutions for this equation.

Keywords Square sum, Smarandache primitive function, equation, solutions.

§1. Introduction and Results

Let p be a prime, n be any positive integer. The Smarandache primitive function $S_p(n)$ is defined as the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_2(1) = 2, S_2(2) = S_2(3) = 4, S_2(4) = 6, \cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Professor Zhang and Liu [2] have studied it, and obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n, we have

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

Li Jie [3] studied the solvability of the equation $S_p(1) + S_p(2) + \cdots + S_p(n) = S_p\left(\frac{n(n+1)}{2}\right)$, and gave all its positive integer solutions. But it seems that no one knows the relationship between the square sum of natural numbers and the Smarandache primitive function. In this paper, we use the elementary methods to study the solvability of the equation

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right),$$

and give all positive integer solutions for it.That is, we will prove the following:

Theorem. Let p be a prime, n be any positive integer. Then the equation

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right) \tag{1}$$

has finite solutions.

(i) If $p = 2, 3$ or 5, then the positive integer solutions of the equation (1) are $n = 1, 2$;

(ii) If $p = 7$, then the positive integer solutions of the equation (1) are $n = 1, 2, 3, 4, 5$;

(iii) If $p \geq 11$, then the positive integer solutions of the equation (1) are $n = 1,\ 2,\ \cdots,\ n_p$, where $n_p \geq 1$ is a positive integer, and

$n_p = \left[\sqrt[3]{\frac{3p}{2} + \sqrt{\frac{9p^2}{4} - \frac{1}{1728}}} + \sqrt[3]{\frac{3p}{2} - \sqrt{\frac{9p^2}{4} - \frac{1}{1728}}} - \frac{1}{2} \right]$, $[x]$ denotes the biggest integer $\leq x$.

§2. Several lemmas

To complete the proof of the theorem, we need the following several simple lemmas.

Lemma 1. Let p be a prime, n be any positive integer, $S_p(n)$denote the Smarandache primitive function, then we have

$$S_p(k) \begin{cases} = pk, & \text{if } \ k \leq p, \\ < pk, & \text{if } \ k > p. \end{cases}$$

Proof. (See reference [4]).

Lemma 2. Let p be a prime, n be any positive integer, if n and p satisfying $p^\alpha \parallel n!$, then

$$\alpha = \sum_{i=1}^{\infty} \left[\frac{n}{p^i} \right].$$

Proof. (See reference [5]).

Lemma 3. Let p be a prime, n be any positive integer. If $n > [\sqrt{p}]$, then there must exist a positive integer m_k with $1 \leq m_k \leq k^2 (k = 1, 2, \cdots, n)$ such that

$$S_p(1^2) = m_1 p, S_p(2^2) = m_2 p, \cdots, S_p(n^2) = m_n p,$$

and

$$k^2 \leq \sum_{i=1}^{\infty} \left[\frac{m_k p}{p^i} \right].$$

Proof. From the definition of $S_p(n)$, Lemma 1 and Lemma 2, we can easily get the conclusions of Lemma 3.

§3. Proof of the theorem

In this section, we will complete the proof of Theorem. We discuss the solutions of the equation $S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right)$ in the following cases:

(I) If $p = 2$, then the equation (1) is $S_2(1^2) + S_2(2^2) + \cdots + S_2(n^2) = S_2\left(\frac{n(n+1)(2n+1)}{6}\right)$.

(a) If $n = 1$, $S_2(1^2) = 2 = S_2\left(\frac{1\times(1+1)\times(2+1)}{6}\right)$, so $n = 1$ is a solution of the equation (1).

(b) If $n = 2$, $S_2(1^2) + S_2(2^2) = 2 + 3\times 2 = 8 = S_2\left(\frac{2\times(2+1)\times(2\times 2+1)}{6}\right)$, so $n = 2$ is a solution of the equation (1).

(c) If $n = 3$, $S_2(1^2) + S_2(2^2) + S_2(2^2) = 2 + 3\times 2 + 6\times 2 = 20$, but $S_2\left(\frac{3\times(3+1)\times(2\times 3+1)}{6}\right) = S_2(14) = 16$, so $n = 3$ is not a solution of the equation (1).

(d) If $n > 3$, then from Lemma 3 we know that there must exist a positive integer m_k with $1 \le m_k \le k^2 (k = 1, 2, \cdots, n)$ such that

$$S_2(1^2) = 2m_1, S_2(2^2) = 2m_2, \ldots, S_2(n^2) = 2m_n.$$

So we have $S_2(1^2) + S_2(2^2) + \cdots + S_2(n^2) = 2(m_1 + m_2 + \cdots + m_n)$.

On the other hand, notice that $m_1 = 1$, $m_2 = 3$, $m_3 = 6$, then from Lemma 3, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n)-1}{2^i}\right]\\
=\ &\sum_{i=1}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n-1)+1}{2^i}\right]\\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{2(m_1+m_2+\cdots+m_n-1)+1}{2^i}\right]\\
=\ &m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{2^i}\right]\\
\ge\ &\left(m_1+\sum_{i=1}^{\infty}\left[\frac{m_1}{2^i}\right]\right)+\left(m_2+\sum_{i=1}^{\infty}\left[\frac{m_2-1}{2^i}\right]\right)+\left((m_3-1)+\sum_{i=1}^{\infty}\left[\frac{m_3}{2^i}\right]\right)\\
&+\left(m_4+\sum_{i=1}^{\infty}\left[\frac{m_4}{2^i}\right]\right)+\cdots+\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{2^i}\right]\right)\\
\ge\ &1^2+2^2+3^2+\sum_{i=1}^{\infty}\left[\frac{2m_4}{2^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{2m_n}{2^i}\right]\\
\ge\ &1^2+2^2+3^2+4^2+\cdots+n^2\\
=\ &\frac{n(n+1)(2n+1)}{6},
\end{aligned}$$

then from Lemma 2, we can get

$$2^{\frac{n(n+1)(2n+1)}{6}} \mid (2(m_1+m_2+\cdots+m_n)-1)!.$$

Therefore,

$$\begin{aligned}
S_2\left(\frac{n(n+1)(2n+1)}{6}\right) &\le 2(m_1+m_2+\cdots+m_n)-1\\
&< 2(m_1+m_2+\cdots+m_n)\\
&= S_2(1^2)+S_2(2^2)+\cdots+S_2(n^2).
\end{aligned}$$

So there is no solutions for the equation (1) in this case.

Hence, if $p = 2$, the equation (1) has only two solutions, they are $n = 1,\ 2$.

If $p = 3$ or 5 using the same method we can easily get if and only if $n = 1,\ 2$ are all solutions of the equation (1).

(II) If $p = 7$, then the equation (1) is

$$S_7(1^2) + S_7(2^2) + \cdots + S_7(n^2) = S_7\left(\frac{n(n+1)(2n+1)}{6}\right).$$

(a) If $n = 1$, $S_7(1^2) = 7 = S_7\left(\frac{1\times(1+1)\times(2+1)}{6}\right)$, so $n = 1$ is a solution of the equation (1).

(b) If $n = 2$, $S_7(1^2) + S_7(2^2) = 7 + 4\times 7 = 35 = S_7\left(\frac{2\times(2+1)\times(2\times 2+1)}{6}\right)$, so $n = 2$ is a solution of the equation (1).

(c) If $n = 3$, $S_7(1^2) + S_7(2^2) + S_7(3^2) = 35 + 8\times 7 = 91 = S_7\left(\frac{3\times(3+1)\times(2\times 3+1)}{6}\right)$, so $n = 3$ is a solution of the equation (1).

(d) If $n = 4$, $S_7(1^2) + S_7(2^2) + S_7(3^2) + S_7(4^2) = 91 + 14\times 7 = 189 =$ $S_7\left(\frac{4\times(4+1)\times(2\times 4+1)}{6}\right)$, so $n = 4$ is a solution of the equation (1).

(e) If $n = 5$, $S_7(1^2) + S_7(2^2) + S_7(3^2) + S_7(4^2) + S_7(5^2) = 189 + 22\times 7 = 343 =$ $S_7\left(\frac{5\times(5+1)\times(2\times 5+1)}{6}\right)$, so $n = 5$ is a solution of the equation (1).

(f) If $n = 6$, $S_7(1^2)+S_7(2^2)+S_7(3^2)+S_7(4^2)+S_7(5^2)+S_7(6^2) = 343+32\times 7 = 567$, but $S_7\left(\frac{6\times(6+1)\times(2\times 6+1)}{6}\right) = S_7(91) = 79\times 7 = 553 < 567$, so $n = 6$ is not a solution of the equation (1).

(g) If $n = 7$, $S_7(1^2)+S_7(2^2)+S_7(3^2)+S_7(4^2)+S_7(5^2)+S_7(6^2)S_7(7^2) = 567+43\times 7 = 868$, but $S_7\left(\frac{7\times(7+1)\times(2\times 7+1)}{6}\right) = S_7(140) = 122\times 7 = 854 < 868$, so $n = 7$ is not a solution of the equation (1).

(h) If $n \geq 8$, from Lemma 3 we know there must be a positive integers m_k with $1 \leq m_k \leq k^2 (k = 1,\ 2,\ \cdots,\ n)$ such that

$$S_7(1^2) = 7m_1, S_7(2^2) = 7m_2, \cdots, S_7(n^2) = 7m_n.$$

Then we have $S_7(1^2) + S_7(2^2) + \cdots + S_7(n^2) = 7(m_1 + m_2 + \cdots + m_n)$.

On the other hand, notice that $m_1 = 1,\ m_2 = 4,\ m_3 = 8,\ m_4 = 14,\ m_5 = 22,\ m_6 =$

32, $m_7 = 43,\ m_8 = 56$, from Lemma 3, we have

$$\begin{aligned}
& \sum_{i=1}^{\infty}\left[\frac{7(m_1+m_2+\cdots+m_n)-1}{7^i}\right] \\
= & \sum_{i=1}^{\infty}\left[\frac{7(m_1+m_2+\cdots+m_n-1)+6}{7^i}\right] \\
= & \ m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{7(m_1+m_2+\cdots+m_n-1)+6}{7^i}\right] \\
= & \ m_1+m_2+\cdots+m_n-1+\sum_{i=1}^{\infty}\left[\frac{m_1+m_2+\cdots+m_n-1}{7^i}\right] \\
\geq & \left(m_1+\sum_{i=1}^{\infty}\left[\frac{m_1}{7^i}\right]\right)+\left(m_2+\sum_{i=1}^{\infty}\left[\frac{m_2}{7^i}\right]\right)+\left(m_3+\sum_{i=1}^{\infty}\left[\frac{m_3-1}{7^i}\right]\right)+\cdots \\
& +\left(m_7+\sum_{i=1}^{\infty}\left[\frac{m_7}{7^i}\right]\right)+\left((m_8-1)+\sum_{i=1}^{\infty}\left[\frac{m_8}{7^i}\right]\right)+\left(m_9+\sum_{i=1}^{\infty}\left[\frac{m_9}{7^i}\right]\right)+\cdots \\
& +\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{7^i}\right]\right) \\
\geq & \ 1^2+2^2+\cdots+8^2+\sum_{i=1}^{\infty}\left[\frac{7m_9}{7^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{7m_n}{7^i}\right] \\
\geq & \ 1^2+2^2+\cdots+n^2 \\
= & \ \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

From Lemma 2, we may immediately get

$$7^{\frac{n(n+1)(2n+1)}{6}} \mid (7(m_1+m_2+\cdots+m_n)-1)!.$$

Therefore,

$$\begin{aligned}
S_7\left(\frac{n(n+1)(2n+1)}{6}\right) & \leq 7(m_1+m_2+\cdots+m_n)-1 \\
& < 7(m_1+m_2+\cdots+m_n) \\
& = S_7(1^2)+S_7(2^2)+\cdots+S_7(n^2).
\end{aligned}$$

So there is no any solutions for the equation (1) in this case.

Hence, if $p = 7$, the equation (1) has only five solutions, they are $n = 1,\ 2,\ 3,\ 4,\ 5$.

(III) If $p \geq 11$ we will discuss the problem in the following cases:

(a) If $\dfrac{n(n+1)(2n+1)}{6} \leq p$, solving this equation we can get $1 \leq n \leq n_p$, and

$$n_p = \left[\sqrt[3]{\frac{3p}{2}+\sqrt{\frac{9p^2}{4}-\frac{1}{1728}}}+\sqrt[3]{\frac{3p}{2}-\sqrt{\frac{9p^2}{4}-\frac{1}{1728}}}-\frac{1}{2}\right],$$

where $[x]$ denotes the biggest integer $\leq x$, then we have

$$S_p\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{n(n+1)(2n+1)}{6}p.$$

Noting that $n_p \leq [\sqrt{p}] < p$, so if $1 \leq n \leq n_p$, then $n^2 \leq p$, from Lemma 1 we have

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = 1^2p + 2^2p + \cdots + n^2p = \frac{n(n+1)(2n+1)}{6}p.$$

Combining above two formulas, we may easily get $n = 1,\ 2,\ \cdots,\ n_p$ are the solutions of the equation $S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = S_p\left(\frac{n(n+1)(2n+1)}{6}\right)$.

(b) If $n_p < n \leq [\sqrt{p}]$, that is $\frac{n(n+1)(2n+1)}{6} > p$ and $n^2 \leq p$, so from Lemma 1 we have

$$S_p\left(\frac{n(n+1)(2n+1)}{6}\right) < \frac{n(n+1)(2n+1)}{6}p,$$

but

$$S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) = 1^2p + 2^2p + \cdots + n^2p = \frac{n(n+1)(2n+1)}{6}p.$$

Hence the equation (1) has no solution in this case.

(c) Let $[\sqrt{p}] = t$, if $n = [\sqrt{p}] + 1 = t + 1$, that is $n^2 > p$, $t \geq 3$. Then

$$\begin{aligned} S_p(1^2) + S_p(2^2) + \cdots + S_p(n^2) &= S_p(1^2) + S_p(2^2) + \cdots + S_p(t^2) + S_p((t+1)^2) \\ &= 1^2p + 2^2p + \cdots + t^2p + (t^2 + 2t)p \\ &= \frac{2t^3 + 9t^2 + 13t}{6}p. \end{aligned}$$

If $p = 11$, that is $n = t + 1 = 4$, so we have $S_{11}(1^2) + S_{11}(2^2) + S_{11}(3^2) + S_{11}(4^2) = \frac{2 \times 3^3 + 9 \times 3^2 + 13 \times 3}{6} \times 11 = 319$, but $S_{11}(\frac{n(n+1)(2n+1)}{6}) = S_{11}(30) = 308 < 319$. So there is no solution for the equation (1) in this case.

If $p = 13$, that is $n = t + 1 = 4$, then we have $S_{13}(1^2) + S_{13}(2^2) + S_{13}(3^2) + S_{13}(4^2) = \frac{2 \times 3^3 + 9 \times 3^2 + 13 \times 3}{6} \times 13 = 377$, but $S_{13}(\frac{n(n+1)(2n+1)}{6}) = S_{13}(30) = 364 < 377$. So there is no solution for the equation (1) in this case.

If $p \geq 17$, that is $t = [\sqrt{p}] \geq 4$, and $n = t + 1 \geq 5$, notice that

$$\begin{aligned} & \sum_{i=1}^{\infty} \left[\frac{\frac{2t^3+9t^2+13t-6}{6}p}{p^i} \right] \\ = \ & \frac{2t^3 + 9t^2 + 13t - 6}{6} + \sum_{i=1}^{\infty} \left[\frac{\frac{2t^3+9t^2+13t-6}{6}}{p^i} \right] \\ \geq \ & \frac{2t^3 + 9t^2 + 13t}{6} - 1 + \sum_{i=1}^{\infty} \left[\frac{\frac{2t^3+9t^2+13t-6}{6}}{p^i} \right] \\ \geq \ & \frac{2t^3 + 9t^2 + 13t}{6} \\ = \ & \frac{(t+1)(t+2)(2t+3)}{6}. \end{aligned}$$

Therefore,

$$\begin{aligned} S_p\left(\frac{n(n+1)(2n+1)}{6}\right) &= S_p\left(\frac{(t+1)(t+2)(2t+3)}{6}\right) \\ &\leq \frac{2t^3+9t^2+13t-6}{6}p \\ &< \frac{2t^3+9t^2+13t}{6}p \\ &= S_p(1^2)+S_p(2^2)+\cdots+S_p(n^2). \end{aligned}$$

So $n=[\sqrt{p}]+1$ is not a solutions of the equation (1).

If $n \geq [\sqrt{p}]+2=t+2$, that is $n^2>p$, $t \geq 3$. Then from Lemma 3, we know that there must exist a positive integers m_k with $1 \leq m_k \leq k^2 (k=1,2,\cdots,n)$ such that

$$S_p(1^2)=m_1p, S_p(2^2)=m_2p, \cdots, S_p(n^2)=m_np,$$

then we have

$$S_p(1^2)+S_p(2^2)+\cdots+S_p(n^2)=(m_1+m_2+\cdots+m_n)p.$$

On the other hand, notice that $m_1=1^2,\ m_2=2^2,\ \cdots,\ m_t=t^2$, from Lemma 3, we have

$$\begin{aligned} & \sum_{i=1}^{\infty}\left[\frac{(m_1+m_2+\cdots+m_n)p-1}{p^i}\right] \\ = & \sum_{i=1}^{\infty}\left[\frac{p(m_1+m_2+\cdots+m_n-1)+p-1}{p^i}\right] \\ = & \ m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{p(m_1+m_2+\cdots+m_n-1)+p-1}{p^i}\right] \\ \geq & \ m_1+m_2+\cdots+m_n-1+\sum_{i=2}^{\infty}\left[\frac{\frac{t(t+1)(2t+1)}{6}p+p-1}{p^i}\right]+ \\ & \sum_{i=1}^{\infty}\left[\frac{m_{t+1}+m_{t+2}+\cdots+m_n-1}{p^i}\right] \\ \geq & \ m_1+m_2+\cdots+m_n+\sum_{i=1}^{\infty}\left[\frac{m_{t+1}+m_{t+2}+\cdots+m_n-1}{p^i}\right] \\ \geq & \ m_1+m_2+\cdots+m_t+\left(m_{t+1}+\sum_{i=1}^{\infty}\left[\frac{m_1}{p^i}\right]\right)+\left(m_{t+2}+\sum_{i=1}^{\infty}\left[\frac{m_{t+2}}{p^i}\right]\right)+\cdots \\ & +\left(m_n+\sum_{i=1}^{\infty}\left[\frac{m_n}{p^i}\right]\right) \\ \geq & \sum_{i=1}^{\infty}\left[\frac{m_1p}{p^i}\right]+\sum_{i=1}^{\infty}\left[\frac{m_2p}{p^i}\right]+\cdots+\sum_{i=1}^{\infty}\left[\frac{m_np}{p^i}\right] \\ \geq & \ 1^2+2^2+\cdots+n^2 \\ = & \ \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Then from Lemma 2, we can get

$$p^{\frac{n(n+1)(2n+1)}{6}} \mid ((m_1+m_2+\cdots+m_n)p-1)!.$$

Therefore,

$$\begin{aligned} S_p\left(\frac{n(n+1)(2n+1)}{6}\right) &\leq (m_1+m_2+\cdots+m_n)p-1 \\ &< (m_1+m_2+\cdots+m_n)p \\ &= S_p(1^2)+S_p(2^2)+\cdots+S_p(n^2). \end{aligned}$$

From the above, we can deduce that if $p \geq 11$ and $n \geq [\sqrt{p}]+2$, then the equation (1) has no solution.

Now the theorem follows from (I), (II) and (III).

References

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