Scientia Magna
Vol. 4 (2008), No. 1, 120-123

On the Smarandache reciprocal function and its mean value

Liping Ding

Department of Mathematics, Xi'an University of Architecture and Technology
Xi'an, 710055, Shaanxi, China

Abstract For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$, and $m+1 \dagger n!$. The main purpose of this paper is using the elementary and analytic methods to study the mean value distribution properties of $S_c(n)$, and give two interesting mean value formulas for it.

Keywords The Smarandache reciprocal function, mean value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m :\ n \mid m!,\ n \in N\}$. And the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$, and $m+1 \dagger n!$. That is, $S_c(n) = \max\{m :\ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } m+1 \dagger n!\}$. The first few values of $S_c(n)$ are:

$$S_c(1) = 1,\ S_c(2) = 2,\ S_c(3) = 3,\ S_c(4) = 4,\ S_c(5) = 6,\ S_c(6) = 6,$$
$$S_c(7) = 10,\ S_c(8) = 10,\ S_c(9) = 10,\ S_c(10) = 10,\ S_c(11) = 12,\ S_c(12) = 12,$$
$$S_c(13) = 16,\ S_c(14) = 16,\ S_5(15) = 16,\ S_c(16) = 16,\ S_c(17) = 18,\ \cdots\cdots.$$

About the properties of $S(n)$, many authors had studied it, and obtained a series results, see references [1], [2], [3], [4], [5] and [15]. For example, Jozsef Sandor [4] proved that for any positive integer $k \geq 2$, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ satisfied the following inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

On the other hand, in reference [6], A.Murthy studied the elementary properties of $S_c(n)$, and proved the following conclusion:

If $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the Smarandache reciprocal function $S_c(n)$, and give two interesting mean value formulas it. That is, we shall prove the following conclusions:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} S_c(n) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

Theorem 2. For any real number $x > 1$, we have the low bound estimate

$$\frac{1}{x}\sum_{n\leq x}\left(S_c(n)-n\right)^2 \geq \frac{1}{3}\cdot \ln^2 x + O\left(x^{-\frac{5}{12}}\cdot \ln^2 x\right).$$

From Theorem 2 we may immediately deduce the following:

Corollary. The limit

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\left(S_c(n)-n\right)^2$$

does not exist.

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. For any real number $x > 1$, let $2 = p_1 < p_2 < \cdots\cdots < p_k \leq x$ denote all primes less than or equal to x. Then from the result of A.Murthy [6] we have the identity

$$\begin{aligned}
\sum_{n\leq x} S_c(n) &= S_c(1)+S_c(2)+S_c(3)+S_c(4)+\sum_{i=3}^{k-1}\sum_{p_i\leq n<p_{i+1}} S_c(n)+\sum_{p_k\leq n\leq x} S_c(n)\\
&= 1+2+3+4+\sum_{i=3}^{k-1}\sum_{p_i\leq n<p_{i+1}}(p_{i+1}-1)+\sum_{p_k\leq n\leq x}(p_{k+1}-1)\\
&= \sum_{i=1}^{k-1}(p_{i+1}-p_i)(p_{i+1}-1)+O\left((x-p_k)(p_{k+1}-1)\right)\\
&= \frac{1}{2}\sum_{i=1}^{k-1}\left[(p_{i+1}-p_i)^2+p_{i+1}^2-p_i^2\right]-\sum_{i=1}^{k-1}(p_{i+1}-p_i)+O\left((x-p_k)\cdot p_{k+1}\right)\\
&= \frac{1}{2}\sum_{i=1}^{k-1}(p_{i+1}-p_i)^2+\frac{1}{2}\cdot p_k^2-p_k+O\left((x-p_k)\cdot p_{k+1}\right). \qquad (1)
\end{aligned}$$

For any real number x large enough, from M.N.Huxley [7] we know that there at least exists a prime in the interval $\left[x,\ x+x^{\frac{7}{12}}\right]$. So we have the estimate

$$(x-p_k)\cdot p_{k+1} = O\left(x^{\frac{19}{12}}\right). \qquad (2)$$

On the other hand, from the D.R.Heath-Brown's famous result [8], [9] and [10] we know that for any real number $\epsilon > 0$, we have the estimate

$$\sum_{i=1}^{k-1}(p_{i+1}-p_i)^2 \ll x^{\frac{23}{18}+\epsilon}. \tag{3}$$

Note that $p_k = x + O\left(x^{\frac{7}{12}}\right)$, from (1), (2) and (3) we may immediately get the asymptotic formula

$$\sum_{n\le x} S_c(n) = \frac{1}{2}\cdot\left[x+O\left(x^{\frac{7}{12}}\right)\right]^2 + O\left(x^{\frac{19}{12}}\right) = \frac{1}{2}\cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

This proves Theorem 1.

Now we prove Theorem 2. For any real number $x > 1$, from the definition and properties of $S_c(n)$ we also have the identity

$$\begin{aligned}
\sum_{n\le x}(S_c(n)-n)^2 &\ge \sum_{i=1}^{k-1}\sum_{p_i\le n<p_{i+1}}(S_c(n)-n)^2 = \sum_{i=3}^{k-1}\sum_{0\le n<p_{i+1}-p_i}(p_{i+1}-p_i-n-1)^2 \\
&= \sum_{i=3}^{k-1}\sum_{0\le n<p_{i+1}-p_i}\left[(p_{i+1}-p_i)^2 - 2(n+1)\cdot(p_{i+1}-p_i) + (n+1)^2\right] \\
&= \sum_{i=3}^{k-1}\left[(p_{i+1}-p_i)^3 - (p_{i+1}-p_i)^2\cdot(p_{i+1}-p_i+1)\right] + \\
&\qquad + \sum_{i=3}^{k-1}\left[\frac{1}{6}\cdot(p_{i+1}-p_i+1)\cdot(p_{i+1}-p_i)\cdot(2p_{i+1}-2p_i+1)\right] \\
&= \frac{1}{3}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3 - \frac{1}{2}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^2 + \frac{1}{6}\sum_{i=3}^{k-1}(p_{i+1}-p_i) \\
&= \frac{1}{3}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3 - \frac{1}{2}\sum_{i=3}^{k-1}(p_{i+1}-p_i)^2 + \frac{1}{6}(p_k-p_3).
\end{aligned} \tag{4}$$

From the Cauchy inequality and the Prime Theorem (see references [11], [12], [13] and [14]) we may get

$$p_k - p_3 = \sum_{i=3}^{k-1}(p_{i+1}-p_i) \le \left(\sum_{i=3}^{k-1}1\right)^{\frac{2}{3}}\left(\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3\right)^{\frac{1}{3}} = (\pi(x)-3)^{\frac{2}{3}}\left(\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3\right)^{\frac{1}{3}}.$$

That is,

$$\left(x+O\left(x^{\frac{7}{12}}\right)\right)^3 = (p_k-p_3)^3 \le \left(\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right)\right)^2\cdot\left(\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3\right)$$

or

$$\sum_{i=3}^{k-1}(p_{i+1}-p_i)^3 \ge x\cdot\ln^2 x + O\left(x^{\frac{7}{12}}\cdot\ln^2 x\right). \tag{5}$$

Combining (4) and (5) we may immediately deduce the low bound estimate

$$\sum_{n\leq x}(S_c(n)-n)^2\geq\frac{1}{3}\cdot x\cdot\ln^2 x+O\left(x^{\frac{7}{12}}\cdot\ln^2 x\right).$$

This completes the proof of Theorem 2.

References

[1] F.Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] Lu Yaming, On the solutions of an equation involving the Smarandache function, Scientia Magna, **2**(2006), No.1, 76-79.

[3] Xu Zhefeng, The value distribution property of the Smarandache function, Acta Mathematica Sinica, Chinese Series, **49**(2006), No.5, 1009-1012.

[4] Jozsef Sandor, On certain inequalities involving the Smarandache function, Scientia Magna, **2**(2006), No.3, 78-80.

[5] Kenichiro Kashihara, Comments and topics on Smarandache notions and problems, Erhus University Press, USA, 1996.

[6] A.Murthy, Smarandache reciprocal function and an elementary inequality, Smarandache Notions Journal, **11**(2000), No. 1-2-3, 312-315.

[7] M.N.Huxley, The distribution of prime numbers, Oxford University Press, Oxford, 1972.

[8] D.R.Heath-Brown, The differences between consecutive primes, Journal of London Mathematical Society, **18**(1978), No.2, 7-13.

[9] D.R.Heath-Brown, The differences between consecutive primes, Journal of London Mathematical Society, **19**(1979), No.2, 207-220.

[10] D.R.Heath-Brown and H.Iwaniec, To the difference of consecutive primes, Invent. Math., **55**(1979), 49-69.

[11] Zhang Wenpeng, The elementary number theory (in Chinese), Shaanxi Normal University Press, Xi'an, 2007.

[12] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.

[13] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem (in Chinese), Shanghai Science and Technology Press, Shanghai, 1988.

[14] Pan Chengdong and Pan Chengbiao, The elementary number theory (in Chinese), Beijing University Press, Beijing, 1992.

[15] I.Balacenoiu and V.Seleacu, History of the Smarandache function, Smarandache Notions Journal, **10**(1999), 192-201.