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Structures of Cycle Bases with Some Extremal Properties

Han Ren and Yun Bai

(Department of Mathematics, East China Normal University, Shanghai 200062, P. R. China)

Email: hren@math.ecnu.edu.cn

Abstract: In this paper, we investigate the structures of cycle bases with extremal properties which are related with *map geometries*, i.e., *Smarandache* 2*-dimensional manifolds*. We first study the long cycle base structures in a cycle space of a graph. Our results show that much information about long cycles is contained in a longest cycle base. (1)Any two longest cycle bases have the same structure, i.e., there is a 1-1 correspondence between any two longest cycle bases such that the corresponding cycles have the same length; (2)Any group of linearly independent longest cycles must be contained in a longest cycle base which implies that any two sets of linearly independent longest cycles with maximum cardinal number is equivalent; (3)If consider the range of embedded graphs, a longest cycle base must contain some long cycles with special properties. As applications, we find explicit formulae for computing longest cycles bases of several class of embedded graphs. As for an embedded graph on non-orientable surfaces, we obtain several interpolation results for one-sided cycles in distinct cycle bases. Similar results for shortest cycle bases may be deduced. For instance, we show that in a strongly embedded graph, there is a cycle base consisting of surface induced non-separating cycles and all of such bases have the same structure provided that their length is of shortest(subject to induced non-separating cycles). These extend Tutte's result [7](which states that in a 3–connected graph the set of induced(graph) non-separating cycles generate the cycle space).

Keywords: Cycle space, longest cycle base, SDR, long cycle.
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§1. Introduction

Here in this paper we consider connected graphs without loops. Concepts and terminologies used without definition may be found in [1]. A spanning subgraph H of G is called an *E-subgraph* iff each vertex has even degree in H. It is well known that the set of E-subgraphs of G forms a linear space $\mathscr{C}(G)$ called the *cycle space* of G. Here, the operation between vectors(i.e., E-subgraphs) is the symmetric difference between edge-sets of E-subgraphs. It is clear that the rank, defined by $\beta(G)$ (the Betti number of G), of $\mathscr{C}(G)$ is $|E(G)| - |V(G)| + 1$ and any set of $\beta(G)$ linearly independence vectors form a base of $\mathscr{C}(G)$. The length $l(\mathcal{B})$ of a cycle base $\mathcal{B}$

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is the sum of length of vectors in it. In particular, the length of an E-subgraph is the sum of length of edge-disjoint cycles in it. Throughout this paper, we only consider the vectors with only one cycle. So, the bases considered are all formed by cycles. By a longest base $\mathcal{B}$ we mean $l(\mathcal{B})$ is the length of a maximum cycle base.

Cycle space theory rooted in early research works of Kirchoff's circuits theory. In theory, Matroid theory is one of motivations of it [10-12], also related with map geometries, i.e., Smarandache 2-dimensional manifolds ([5]-[6]). In particular, cycle bases with minimum length have many applications in structural analysis [2], chemical storage theory [3], as well as fields such bioscience [4]. In history, classical works concentrated on minimum cycle bases(i.e., MCB). On the other direction, results for cycle spaces theory on long cycles are seldom to be seen. What can we say about longest cycle bases? In intuition, a longest cycle base should contain information about long cycles(especially the longest cycles). Here, in this paper we investigate the structure of longest cycle bases. Based on a Hall type theorem for base transformation, we present a condition for a cycle base to be longest.

Theorem A *Let $\mathcal{B}$ be a cycle base(i.e., vectors of $\mathcal{B}$ are all cycles) of G. Then $\mathcal{B}$ is longest if and only if for every cycle C of G:*

$$\forall\, \alpha \in \mathrm{Int}(C) \Longrightarrow |\,\alpha\,| \geq |\,C\,| \tag{1}$$

where $\mathrm{Int}(C)$ *is the set of cycles in $\mathcal{B}$ which span C.*

Note: (1) This condition says that for a longest base $\mathcal{B}$, any cycle can't be generated by shorter cycles of $\mathcal{B}$;

(2) One may see that such Hall type theorem is very useful in studies of cycle bases with particular extremal properties.

The following result shows that any group of linearly independent longest cycles are contained in a longest cycle base. In particular, any longest cycle is contained in a longest cycle base.

Theorem B *Let $C_1, C_2, \ldots, C_s$ be a set of linearly independent longest cycles of graph G. Then there is a longest cycle base $\mathcal{B}$ containing $C_i, 1 \leq i \leq s$.*

If consider the cycles passing through an edge, then after using Theorem A we may see that for every edge e of a graph G, every longest cycle base must contain a cycle which is longest among cycles passing through e.

Corollary 1 *Any longest cycle of a graph must be contained in a longest cycle base.*

Based on Theorem A, we obtain the following unique structure of longest cycle bases.

Theorem C *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of two longest cycle bases of a graph G. Then there is a* 1-1 *correspondence φ between $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for each cycle $\alpha \in \mathcal{B}_1$, $|\,\varphi(\alpha)| = |\,\alpha|$.*

Corollary 2 *A graph G's any two longest cycle bases must contain the same number of k-cycles, for k = 3, 4, $\ldots$,n.*

Since the condition (1) of Theorem A implies that a cycle can't be generated by shorter cycles in a longest cycle base, we have the following

Corollary 3 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of two longest cycle bases of a graph G. Then the two subgroups of $\mathcal{B}_1$ and $\mathcal{B}_2$ which contain longest cycles are linearly equivalent.*

Corollary 4 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of two longest cycle bases of a graph G and $A_k, A_k^{'}$ be the sets of k-cycles of $\mathcal{B}_1$ and $\mathcal{B}_2$, resp. Then $\bigcup\limits_{k=p}^{n} A_k$ is equivalent to $\bigcup\limits_{k=p}^{n} A_k^{'}$, for each p = 3, 4, $\ldots$,n.*

As applications of Theorems A-C, we will compute the length of longest cycle bases in several types of graphs. But what surprises us most is that those results are also very useful in computing cycle bases with particular extremal properties. In particular, we have the following

Theorem D *Let G be an embedded graph with $\mathcal{B}_1$ and $\mathcal{B}_2$ to be a pair of its longest(shortest) cycles bases. If $\mathcal{B}_1$ and $\mathcal{B}_2$ contain, resp., s and t distinct one-sided cycles, then there is a longest(shortest) cycle base $\mathcal{B}$ with exactly k distinct one-sided cycles for every integer k between s and t.*

Since our results may be applied to any pair of bases, we have

Theorem D$^{'}$ *Let G be an embedded graph, and $\mathcal{B}_1, \mathcal{B}_2$ be a pair of cycle bases containing, resp., m and n one-sided cycles. Then G has a cycle base containing exactly k distinct one-sided cycles for any natural number k between m and n.*

A cycle C of an embedded graph G in a surface $\sum$ is called *(surface)non-separating* if $\sum - C$ is connected; otherwise, it is *(surface)separating*. If one component of $\sum - C$ is an open disc, then C is *contractible* or *trivial*; if not so, C is called *non-contractible*. It is clear that a non-separating cycle is also non-contractible. Since a non-separating cycle can't be spanned by separating cycles (as we will show later), we have the following result.

Theorem E *A longest cycle base of an embedded graph must contain a longest non-separating cycle; any longest non-separating cycle is also contained in a longest cycle base; furthermore, if a pair of longest cycle bases contains, respectively, m and n longest non-separating cycles, then for every integer $k : m \leq k \leq n$, there is a longest cycle base containing exactly k longest non-separating cycles .*

On the other direction, if we consider the shortest cycle bases, then interesting properties on short cycles will appear. We call a graph G in a surface to be *LEW-embedded* if the length of shortest non-contractible cycle is longer than any facial walk. It is well known that an LEW-embedded graph shares many properties with planar graphs [8]. Here, we will present some more unknown results for cycle bases of LEW-embedded graphs.

Theorem F *Let G be an LEW-embedded graph and $\mathcal{B}_1, \mathcal{B}_2$ be a pair of shortest cycle bases. Then, we have the following results:*
(1) For any separating cycle $C \in \mathcal{B}_i$ and non-separating cycle $C^{'} \in \mathcal{B}_i$, $|C^{'}| > |C|$;
(2) Both $\mathcal{B}_1$ and $\mathcal{B}_2$ contain exactly $\nu(\sum)$ non-separating cycles, where $\nu(\sum)$ is the Euler-genus of the surface $\sum$ in which G is embedded; further more, the subsets of separating cycles of $\mathcal{B}_1$

and $\mathcal{B}_2$ are linearly equivalent;
(3) Both $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same number of shortest non-separating cycles.

If we restrict some condition on an embedded graph, then some unknown results are obtained. For instance, we have the following

Theorem G *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of longest cycle bases of an embedded graph G. If the length of longest non-separating cycle is longer than that of any separating cycle, then both $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same number of longest non-separating cycles.*

A cycle of a graph is *induced* if it has no chord. A famous result in cycle space theory is due to W. Tutte which states that in a simple 3–connected graph, the set of induced cycles each of which can't separate the graph generates the whole cycle space [9]. If we consider the case of embedded graphs, then this cycle set may be smaller. In fact, we have the following

Theorem *Let G be a 2–connected graph embedded in a non-spherical surface such that its facial walks are all cycles. Then there is a cycle base consists of induced non-separating cycles.*

Remark(1) Tutte's definition of a non-separating cycle differs from ours. The former defined a cycle which can't separate the graph, while the latter define a cycle which can't separate the surface in which the graph is embedded. So, Theorem H and Tutte's result are different. From our proof one may see that this base is determined simply by (surface)non-separating cycles. As for the structure of such bases, we may modify the condition of Theorem A and obtain another condition for bases consisting of shortest non-separating cycles.

Remark(2) Theorem H implies the existence of a cycle base $\mathcal{B}$ satisfying

i) All cycles in this cycle base $\mathcal{B}$ are non-separating;
ii) The length of this base $\mathcal{B}$ is shortest subject to i).

We call a base defined above as *shortest non-separating cycle base.*

Theorem I *Let G be a 2–connected graph embedded in a non-spherical surface such that all of its facial walks are cycles. Let $\mathcal{B}$ be a base consisting of non-separating cycles. Then $\mathcal{B}$ is shortest iff for every non-separating cycle C,*

$$\forall\ \alpha \in \mathrm{Int}(C) \Rightarrow |\,C| \geq |\,\alpha|,$$

where $\mathrm{Int}(C)$ *is the subset of cycles of $\mathcal{B}$ which span C.*

Combining Theorems H and I we obtain the following unique structure result for shortest non-separating cycle bases.

Theorem J *Let G be a 2–connected graph embedded in some non-spherical surface with all its facial walks as cycles. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be a pair of shortest non-separating cycle bases. Then there exists a 1–1 correspondence φ between elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for every element $\alpha \in \mathcal{B}_1$, $|\,\alpha| = |\,\varphi(\alpha)|$.*

Remark From our proof of Theorem J, one may see that if the surface in which the graph is embedded is non-orientable, then we may find a cycle base consisting of one–sided cycles and so, there is a cycle base satisfying

i) All cycles in the base are one–sided cycles;

ii) The length of the base is shortest subject to i);

iii) Any pair of cycle bases satisfying i) and ii) have the same structure, i.e., there is a 1–1 correspondence between them such that the corresponding cycles have the same length.

§2. Proofs of general results

In this section we shall prove Theorems A – C. Firstly, we should set up some preliminaries works. Let $\mathcal{M} = (S_1, S_2, \ldots, S_m)$ be a set of m sets. If each S_i contains an element a_i such that $a_i \neq a_j$ for $i \neq j$, then $(a_1, a_2, \ldots, a_m)$ is called a SDR of $\mathcal{M}$. The following is a famous condition for a system of sets to have a SDR.

Lemma 1(Hall's theorem [7]) *Let $\mathcal{M} = (S_1, S_2, \ldots, S_m)$ be a system of sets. Then $\mathcal{M}$ has a SDR iff for any k subsets of $\mathcal{M}$, their union has at least k elements,* $1 \leq k \leq m$.

The following is an application of Lemma 1.

Lemma 2 *Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}, \mathcal{B}_2 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be a pair of bases of a linearly vector space $\mathcal{V}_m$ over a field $\mathscr{F}$. Then $\mathcal{M} = (S_1, S_2, \ldots, S_m)$ has a SDR, where $S_i = \mathrm{Int}(\alpha_i)$ is the set of vectors of $\mathcal{B}_2$ which spans $\alpha_i, 1 \leq i \leq m$.*

Proof Suppose on the contrary. Then there is an integer number k and k subsets, say $S_1, S_2, \ldots, S_k$ such that

$$\left|\bigcup_{i=1}^{k} S_i\right| < k \tag{2}$$

This shows that $\alpha_1, \alpha_2, \ldots, \alpha_k$ may be generated by less than k elements of $\mathcal{B}_2$, a contradiction as desired. □

Proof of Theorem A Let $\mathcal{B}$ be a longest cycle base of G and C be a cycle of G. Then there is a set $\mathrm{Int}(C)$ of cycles of $\mathcal{B}$ which span C, i.e., $C = \sum\limits_{C_i \in \mathrm{C}} \oplus\, C_i$. If there is a cycle $C_i \in \mathrm{Int}(C)$ with $|\,C_i| < |\,C|$, then $\mathcal{B}_1 = \mathcal{B} - C_i + C$ is another cycle base with length longer than that of $\mathcal{B}_1$, contrary to the definition of $\mathcal{B}$. Thus, (1) holds for every cycle of G. On the other hand, suppose that $\mathcal{B}$ is a cycle base of G satisfying (1) and $\mathcal{B}_1$ is a longest cycle base of G. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_m\},\ \mathcal{B}_1 = \{\gamma_1, \gamma_2, \ldots, \gamma_m\},\ m = \beta(G)$. Then for each $\gamma_i \in \mathcal{B}_1$, there is a set $\mathrm{Int}(\gamma_i)$ of cycles of $\mathcal{B}$ which span γ_i. By Lemma 2, $(\mathrm{Int}(\gamma_1), \mathrm{Int}(\gamma_2), \ldots, \mathrm{Int}(\gamma_m))$ has a SDR$= (\alpha_1^{'}, \alpha_2^{'}, \ldots, \alpha_m^{'})$ such that $\alpha_i^{'} \in \mathrm{Int}(\gamma_i), 1 \leq i \leq m$. Then by (1), we have that

$$|\,\alpha_i^{'}| \geq |\,\gamma_i|\,, \quad 1 \leq i \leq m$$

which implies that $l(\mathcal{B}) \geq l(\mathcal{B}_1)$ and so, $\mathcal{B}$ is also a longest cycle base of G. □

Proof of Theorem B Let $\mathcal{B}$ be a longest cycle base of G such that $|\,\mathcal{B} \cap \{C_1, C_2, \ldots, C_s\}|$ is as large as possible. If $|\,\mathcal{B} \cap \{C_1, C_2, \ldots, C_s\}| = s$, then $C_i \in \mathcal{B}$ for $1 \leq i \leq s$. $\mathcal{B}$ is the right cycle base. Otherwise, there is an integer $k\,(1 \leq k \leq s)$ such that $C_k \notin \mathcal{B}$. Then $\mathcal{B}$ has a subset $\mathrm{Int}(C_k)$ spanning C_k. It is clear that $\mathrm{Int}(C_k) \nsubseteq \{C_1, C_2, \ldots, C_s\}$. Hence, there is a cycle $C_j \in \mathrm{Int}(C_k)\backslash\{C_1, C_2, \ldots, C_s\}$. Since Theorem A shows that a cycle can't be generated by shorter cycles in a longest cycle base, we have that $|C_j| = |C_k|$. Thus, $\mathcal{B}_1 = \mathcal{B} - C_j + C_k$ is a longest cycle base containing more cycles in $\{C_1, C_2, \ldots, C_s\}$ than that of $\mathcal{B}$, a contradiction as desired. $\square$

Proof of Theorem C Let $\mathcal{B}_1 = \{C_1, C_2, \ldots, C_m\}$, $\mathcal{B}_2 = \{C_1^{'}, C_2^{'}, \ldots, C_m^{'}\}$ be a pair of longest cycle bases of G, $m = \beta(G)$. Then for each $C_i^{'} \in \mathcal{B}_2$, there is a subset $\mathrm{Int}(C_i^{'}) \subseteq \mathcal{B}_1$ such that $C_i^{'}$ is spanned by vectors of $\mathrm{Int}(C_i^{'})$. By Lemma 2, $\left(\mathrm{Int}(C_1^{'}), \mathrm{Int}(C_2^{'}), \ldots, \mathrm{Int}(C_m^{'})\right)$ has a SDR, say $(C_1, C_2, \ldots, C_m)$ with $C_i \in \mathrm{Int}(C_i^{'})$, $1 \leq i \leq m$. By Theorem A, $|C_i^{'}| \leq |C_i|$, $1 \leq i \leq m$. Let $\varphi : C_i \longmapsto C_i^{'}$. Then φ is a 1-1 correspondence between $\mathcal{B}_1$ and $\mathcal{B}_2$. Since both of them are longest, we have that $|\varphi(C_i)| = |C_i^{'}| = |C_i|$, $1 \leq i \leq m$. This ends the proof of Theorem C. $\square$

§3. Applications to embedded Graphs

In this section, we shall apply the results of § 2 to obtain some important results in graph theory. We first introduce some definition for graph embedding. Let G be a graph which is topologically embedded in a surface S such that each component of $S - G$ is an open disc. Such graph embedding are called *2-cell embedding*. We may also define such embedding in another way as the monograph [8] did. An embedding of a graph is a rotations system $\pi = \{\pi_v | \, v \in V(G)\}$ (each π_v is a cyclic permutation of semi-edges around v) with a signature $\pi : E(G) \longmapsto \{-1, 1\}$. If a cycle C has even-number of negative signatures, it is called a *two-sided cycle*; otherwise, it is called a *one-sided cycle*. If an embedding permits no one-sided cycles, then it is called an *orientable embedding*; otherwise, it is *non-orientable embedding*. It is clear that a one-sided cycle is contained in a Möbius band which bounds a crosscap.

Proof of Theorem D Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $\mathcal{B}_2 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be a pair of longest(shortest) cycle bases of a graph G, $m = \beta(G)$, such that $\mathcal{B}_1$ and $\mathcal{B}_2$ have s and t one-sided cycles, resp. Suppose that $s < t$ and k is an integer : $s \leq k \leq t$. We will show that there exists a longest cycle base $\mathcal{B}$ with exactly k one-sided cycles. We apply induction on the value of $|s - t|$. It is clear that the result holds for smaller value. Now suppose that it holds for values smaller than $|s - t|$. By Lemma 2, $(\mathrm{Int}(\beta_1), \mathrm{Int}(\beta_2), \ldots, \mathrm{Int}(\beta_m))$ has a SDR, say $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m})$ with $\alpha_{i_j} \in \mathrm{Int}(\beta_j)$, where each $\mathrm{Int}(\beta_j)$ is the set of cycles of $\mathcal{B}_1$ which span $\beta_j, 1 \leq j \leq m$. Further, $|\alpha_{i_j}| = |\beta_j|$ by the definition of $\mathcal{B}_1$ and $\mathcal{B}_2, 1 \leq j \leq m$. Since $\mathcal{B}_2$ has more one-sided cycles than that of $\mathcal{B}_1$, there is a one-sided cycle β_j such that $\mathrm{Int}(\beta_j)$ contains a two-sided cycle, say $\alpha_j^{'}$, of $\mathcal{B}_1$. In fact, we may choose $\alpha_{i_j} = \alpha_j^{'}$ by the 1-1 correspondence. Now let $\mathcal{B} = \mathcal{B}_1 - \alpha_{i_j} + \beta_j$. Then $\mathcal{B}$ is another longest cycle base with exactly $s + 1$ one-sided cycles. By induction hypothesis, the result holds. $\square$

Proof of Theorem D$'$ It follows from the proof of Theorem D. $\square$

Before our proving of Theorem E, we have to do some preliminary works. First, we have the following result for surface topology.

Lemma 3 *Let G be an embedded graph and C a non-separating cycle of G. Then C can't be* *generated by a group of separating cycles.*

Proof Since every separating cycle is two–sided and a one–sided cycle can't be spanned by two–sided cycles, we may suppose that C is a two-sided non-separating cycle. Recall that C is non-separating iff $G_l(C) = G_r(C)$, where $G_l(C)$ and $G_r(C)$ are, respectively, the left–subgraph and right–subgraph of C (as defined in [8]). Suppose that C may be spanned by a set of separating cycles. Then C may also by spanned by a set of facial walks : $\partial f_1, \partial f_2, \ldots, \partial f_s$, i.e.,

$$C = \partial f_1 \oplus \partial f_2 \oplus \ldots \oplus \partial f_s\,, \qquad \mathrm{Int}(C) = \{\,\partial f_1, \partial f_2, \ldots, \partial f_s\,\},$$

This implies that for every edge e of C, e is covered(contained) in exactly one facial walk of $\mathrm{Int}(C) = \{\,\partial f_1, \partial f_2, \ldots, \partial f_s\,\}$ and every edge in $\{\,\partial f_1, \partial f_2, \ldots, \partial f_s\,\}\backslash E(C)$ is contained in exactly two walks in $\{\,\partial f_1, \partial f_2, \ldots, \partial f_s\,\}$.

Let $x \in V(C)$ and e be an edge of C containing x. Then the local rotation of edges incident to x is $\Pi_x = (\,e, e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_q)$, where e_{p+1} is another edge of C having a common vertex with e. Each pair of consecutive edges forms a corner $\angle e_i\, x e_{i+1}$ containing x. It is clear that each corner is contained in a region bounded by some facial walk in $\mathrm{Int}(C)$. If the corner $\angle exe_1$ is contained in a region bounded by a facial walk, then each corner $\angle e_i\, x e_{i+1}\,(1 \le i \le p)$ is also contained in some facial walk. In particular, e_{p+1} is also contained in a facial walk. Thus, if a facial walk of $\mathrm{Int}(C)$ is on the right–hand side of C and shares an edge with C, then all corner together with its edges on the right–side of C are contained in facial walks of $\mathrm{Int}(C)$. Since each edge of C is contained in exactly one facial walk of $\mathrm{Int}(C)$, we see that no facial walk of $\mathrm{Int}(C)$ may contain an edge of C which is in $G_l(C)$. Notice that C is non-separating and thus there is an path P starting from an edge of $G_r(C)$ containing a vertex of C and ending at another edge in $G_l(C)$ which contains a vertex of C. This implies that G^*, the dual graph of G, contains a path P^* connecting a pair of facial walks which are on the distinct side of C. We may choose P^* such that it has no edge corresponding to an edge of C. It is easy to see that the vertices of P^* correspond to a set of facial walks of $\mathrm{Int}(C)$ which form a facial walk chain. Hence, the two end–facial walks corresponding to the two end–vertices of P must be in $\mathrm{Int}(C)$. This is impossible since $\mathrm{Int}(C)$ has no such pair of facial walks (containing edges in C) on distinct side of C. This ends the proof of Lemma 3. □

Proof of Theorem E Let $\mathcal{B}$ be a longest cycle base and C a longest non-separating cycle. If $C \notin \mathcal{B}$, then C is spanned by a set $\mathrm{Int}(C)$ of cycles of $\mathcal{B}$. By Lemma 3, $\mathrm{Int}(C)$ contains a non-separating cycle $C^{'}$ which is no shorter than that of C (by (1) of Theorem A), so $|\,C| = |\,C^{'}|$ and $C^{'}$ is also a longest non-separating cycle. This proves the first part of Theorem E. Now let

$$\begin{aligned}\mathcal{B}_1 &= \{\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_{\beta(G)}\},\\ \mathcal{B}_2 &= \{\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_{\beta(G)}\},\end{aligned}$$

be a pair of longest cycle bases with exactly m and n non-separating cycles. Let $\alpha_i(1 \le i \le m)$ and $\gamma_j(1 \le j \le n)$ be non-separating cycles of $\mathcal{B}_1$ and $\mathcal{B}_1$, respectively. Then for each $\gamma_i \in \mathcal{B}_2$, there is a set $\mathrm{Int}(\gamma_i)$ of cycles of $\mathcal{B}_1$ spanning γ_i. By the proving procedure of Theorem A, the system of sets

$$\big(\mathrm{Int}(\gamma_1), \mathrm{Int}(\gamma_2), \ldots, \mathrm{Int}(\gamma_n), \ldots, \mathrm{Int}(\gamma_{\beta(G)})\big)$$

has a SDR $(\alpha_1', \alpha_2', \ldots, \alpha_n', \ldots, \alpha_{\beta(G)}')$ and further $\alpha_i' \in \mathrm{Int}(\gamma_i)$ such that $|\alpha_i'| = |\gamma_i|, 1 \le i \le \beta(G)$. It is clear that there is an integer, say $k(1 \le k \le n)$, such that α_k' is separating since $m < n$ implies that $\mathcal{B}_2$ has more longest non-separating cycle than that of $\mathcal{B}_1$. Now consider the set $\mathcal{B}_3 = \mathcal{B}_2 - \gamma_k + \alpha_k'$ is a longest cycle base containing exactly $n-1$ longest non-separating cycles. Repeating this procedure, we may find a longest cycle base with exactly l longest non-separating cycles for each $l : m \le l \le n$. This ends the proof of Theorem E. $\square$

Proof of Theorem F Let $\mathcal{B}_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_{\beta(G)}\}$ be a MCB (minimum cycle base) of an LEW-embedded graph G, where $\alpha_i(1 \le i \le m)$ and $\alpha_j(m < j \le \beta(G))$ are, respectively, non-separating cycle and separating cycle. Suppose that there are φ facial walks: $\partial f_1, \partial f_2, \ldots, \partial f_\varphi$. It is clear that $\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_{\beta(G)}$ may be linearly expressed by $\{\partial f_1, \partial f_2, \ldots, \partial f_{\varphi-1}\}$. Let $\partial f_i(1 \le i \le \varphi - 1)$ be a facial walk. Then ∂f_i is spanned by a subset $\mathrm{Int}(\partial f_i)$ of $\mathcal{B}_1$. Since $\mathcal{B}_1$ is shortest, every cycle of $\mathrm{Int}(\partial f_i)$ must be contractible by Theorem A. Thus, $\{\partial f_1, \partial f_2, \ldots, \partial f_{\varphi-1}\}$ is linearly equivalent to $\{\alpha_{m+1}, \alpha_{m+2}, \ldots, \alpha_{\beta(G)}\}$, i.e., $\beta(G) - m = \varphi - 1$(which says that $\mathcal{B}_1$ has exactly $\nu(\sum)$ non-separating cycles, where $\nu(\sum)$ is the Euler-genus of the host surface $\sum$ on which G is embedded). This ends the proof of (2).

Let α_i and α_j be, respectively, non-separating cycle and separating cycle of $\mathcal{B}_1$ such that $|\alpha_i| \le |\alpha_j|$. Then α_j is spanned by a set $\mathrm{Int}(\alpha_j)$ of facial walks. It is clear that there is a facial walk, say α_k, of $\mathrm{Int}(\alpha_j)$ which can't be generated by vectors in $\mathcal{B}_1 \backslash \{\alpha_j\}$. It is easy to see that $|\partial f_k| < |\alpha_j|$(since otherwise, $|\alpha_i| \le |\partial f_k|$ will contrary to the definition of LEW-embedded graph). Hence, $\mathcal{B}_1 - \alpha_j + \partial f_k$ will be a shorter cycle base, contrary to the definition of $\mathcal{B}_1$. So, we have $|\alpha_i| > |\alpha_j|$ which ends the proof of (1).

Let

$$\begin{aligned} \mathcal{B}_1 &= \{\alpha_1, \alpha_2, \ldots, \alpha_s, \alpha_{s+1}, \ldots, \alpha_{\beta(G)}\}, \\ \mathcal{B}_2 &= \{\gamma_1, \gamma_2, \ldots, \gamma_t, \gamma_{t+1}, \ldots, \gamma_{\beta(G)}\}, \end{aligned}$$

be a pair of MCBs such that $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ and $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ are, respectively, the set of longest non-separating cycles of $\mathcal{B}_1$ and $\mathcal{B}_2$. Suppose that $s \le t$. Then for each $\gamma_i(1 \le i \le \beta(G))$, there is a subset $\mathrm{Int}(\gamma_i)$ of $\mathcal{B}_1$ which span γ_i. By the proving procedure of Theorem A, the system of sets: $(\mathrm{Int}(\gamma_1), \mathrm{Int}(\gamma_2), \ldots, \mathrm{Int}(\gamma_{\beta(G)}))$ has a SDR, say $(\alpha_1', \alpha_2', \ldots, \alpha_{\beta(G)}')$ such that $\alpha_i' \in \mathrm{Int}(\gamma_i)$ and $|\alpha_i'| = |\gamma_i|, 1 \le i \le \beta(G)$. By (1) we see that each $\alpha_i'(1 \le i \le t)$ is non-separating which implies that $\alpha_1', \alpha_2', \ldots, \alpha_t'$ is a collection of longest non-separating cycles of G in $\mathcal{B}_1$. Thus, $t \le s$. This ends the proof of (3). $\square$

Proof of Theorem G It follows from the proving procedure of Theorem E. $\square$

Proof of Theorem H Notice that any cycle base consists of two parts: the first part is determined by non-separating cycles while the second part is composed of separating cycles. So, what we have to do is to show that any facial cycle may be generated by non-separating cycles. Our proof depends on two steps.

Step 1 Let x be a vertex of G. Then there is a non-separating cycle passing through x.

Let $C^{'}$ be a non-separating cycle of G which avoids x. Then by Menger's theorem, there are two inner disjoint paths P_1 and P_2 connecting x and $C^{'}$. Let $P_1 \cap C^{'} = \{u\}$, $P_2 \cap C^{'} = \{v\}$. Suppose further that $u\overrightarrow{C^{'}}v$ and $v\overrightarrow{C^{'}}u$ are two segments of $C^{'}$, where $\overrightarrow{C}$ is an orientation of C. Then there are three inner disjoint paths connecting u and v:

$$Q_1 = u\overrightarrow{C}v, \quad Q_2 = v\overrightarrow{C}u, \quad Q_3 = P_1 \cup P_2\,.$$

Since $C^{'} = Q_1 \cup Q_2$ is non-separating, at least one of cycles $Q_2 \cup Q_3$ and $Q_1 \cup Q_3$ is non-separating by Lemma 3.

Step 2 Let ∂f be any facial cycle. Then there exist two non-separating cycles C_1 and C_2 which span ∂f.

In fact, we add a new vertex x into the inner region of $\partial f(i.e., int(\partial f))$ and join new edges to each vertex of ∂f. Then the resulting graph also satisfies the condition of Theorem H. By Step 1, there is a non-separating C passing through x. Let u and v be two vertices of $C \cap \partial f$. Then $u\overrightarrow{C}v$ together with two segments of ∂f connecting u and v forms a pair of non-separating cycles. □

Proof of Theorem I and J It follows from the proving procedure of Theorem A and C. □

§4. Examples

Next, we will compute the lengths of longest cycle bases in some types of graphs.

Example 1 Let G be a " Möbius ladder graph " embedded in the projective plane as shown in Fig.1.

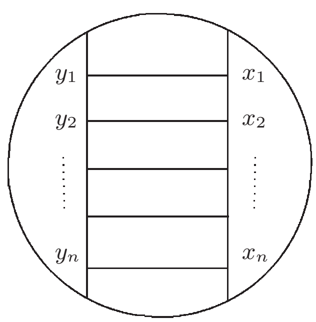

Fig. 1

It is clear that G is non-planar and 3-regular. There are n quadrangles defined as

$$C_4^{(i)} = \begin{cases} (\,x_i, x_{i+1}, y_{i+1}, y_i)\,, & 1 \leq i \leq n-1 \\ \\ (\,x_n, y_1, x_1, y_n)\,, & i = n \end{cases}$$

and n Hamiltonian cycles as

$$H_i = \begin{cases} H - \{(x_i, x_{i+1}), (y_i, y_{i+1})\} + \{(x_i, y_i), (x_{i+1}, y_{i+1})\}, & 1 \le i \le n-1 \\ \\ (x_1, x_2, \dots, x_n, y_n, y_{n-1}, \dots, y_2, y_1), & i = n \end{cases}$$

where H is the Hamiltonian cycle $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$. It is easy to see that $C_4^{(i)} \oplus H_i$ is the Hamiltonian cycle H.

Case 1 $n \equiv 0\,(mod\ 2)$.

Claim 1 $\{H_1, H_2, \dots, H_n\}$ *is a linearly independent set.*

If not so, one may see that

$$H_1 \oplus H_2 \oplus \cdots \oplus H_n = 0$$

This implies that

$$\left(H_1 \oplus C_4^1\right) \oplus \left(H_2 \oplus C_4^2\right) \oplus \cdots \oplus \left(H_n \oplus C_4^n\right) = C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^n$$

i.e.,

$$n\,H = H = 0\,,$$

a contradiction.

Let C be a (2n-1)-cycle which is non-contractible. Since $n \equiv 0\,(mod\ 2)$, we have

Claim 2 *C can't be generated by* $\{H_1, H_2, \dots, H_n\}$.

This follows from the fact that C is a one-sided cycle which can't be spanned by two-sided cycles. Then $\mathcal{B} = \{C, H_1, H_2, \dots, H_n\}$ is a longest cycle base. Otherwise, G would have a longest cycle base which consists of $n+1$ Hamiltonian cycles, and so G is bipartite. This is a contradiction with the fact that G has an odd cycle $(x_1, x_2, \dots, x_n, y_n)$.

Case 2 $n \equiv 1\,(mod\ 2)$

Claim 3 $\{H_1, H_2, \dots, H_{n-1}\}$ *is a set of linearly independent cycles.*

This time, we consider the contractible Hamiltonian cycle H. Then $\{H_1, H_2, \dots, H_{n-1}, H\}$ is also a set of linearly independent cycles. If not so, H would be the sum of $H_1, H_2, \dots, H_{n-1}$, i.e.,

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_{n-1}\,,$$

that is,

$$\begin{aligned} H \oplus C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^{n-1} &= \left(H_1 \oplus C_4^1\right) \oplus \left(H_2 \oplus C_4^2\right) \oplus \left(H_{n-1} \oplus C_4^{n-1}\right) \\ &= (n-1)\,H = \ 0\,, \end{aligned}$$

Now, we have that

$$H = C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^{n-1}\,.$$

This is impossible(since $C_4^1 \oplus C_4^2 \oplus \cdots \oplus C_4^{n-1} \oplus C_4^n = H$).

Let $H^{'}$ be a non-contractible Hamiltonian cycle. Then by Claim 2, $\mathcal{B} = \{H_1, H_2, \ldots, H_{n-1}, H, H^{'}\}$ is a Hamiltonian base of G.

Example 2 Let us consider the longest cycle base of $\mathcal{K}_n$, the complete graph with n vertices. It is easy to see that $\beta(\mathcal{K}_n) = \frac{1}{2}(n-1)(n-2) = C_{n-1}^{\,2}$, which suggests us to give a combinatorial explanation of $\beta(\mathcal{K}_n)$. Suppose $V(G) = \{x_1, x_2, \ldots, x_n\}$. Then $\mathcal{K}_n - x_n = \mathcal{K}_{n-1}$, i.e., the complete graph with $n-1$ vertices $x_1, x_2, \ldots, x_{n-1}$. Let us consider a (n-1)-cycle $\overrightarrow{C}_{n-1} = (x_1, x_2, \ldots, x_{n-1})$ and $x_i,\ x_j \in V(C_{n-1})(i < j)$. Then $H_{i,j} = x_{i-1}\overleftarrow{C}C_{n-1}x_j\, x_i\overrightarrow{C}_{n-1}x_{j-1}$ is a Hamiltonian path of $\mathcal{K}_{n-1}$. Now we find $\beta(\mathcal{K}_n)$ Hamiltonian cycles defined as $C_n(i,\, j) = (x_n\, x_{i-1}\overleftarrow{C}C_{n-1}x_j\, x_i\overrightarrow{C}_{n-1}x_{j-1})$ in formal.

Claim 4 *If* $|\, i - j| \geq 2$, *then the set* $\{C_n(i,\, j) | 1 \leq i < j \leq n-1\}$ *is linearly independent set.*

This follows frow the fact that $(x_i,\, x_j) \in E(C_n(i,\, j))$ is an edge which can't be deleted by the definition of symmetric difference.

Case 1 $n \equiv 1\, (mod\ 2)$

Now the n -cycles $C_n(i,\, i+1) = (x_n, x_{i+1}, x_{i+2}, \ldots, x_{n-1}, x_1, x_2, \ldots, x_i), (1 \leq i \leq n-1)$ is linearly independent cycles. Otherwise, we have that

$$C_n(1,\, 2) \oplus C_n(2,\, 3) \oplus \ldots \oplus C_n(n-1,\, 1) = 0$$

which implies $\cap C_{n-1} = 0$, a contradiction! Based on this and Claim 4, $\{C_n(i,\, j) | 1 \leq i < j \leq n-1\}$ is a set of linearly independent Hamiltonian cycles.

Case 2 $n \equiv 0\, (mod\ 2)$

Although $\{C_n(1,\, 2), C_n(2,\, 3), \ldots, C_n(n,\, 1)\}$ is linearly dependent set of Hamilton cycles, $\{C_n(1,\, 2), C_n(2,\, 3), \ldots, C_n(n-1,\, n)\}$ is a set of linearly independent cycles. Since $\mathcal{K}_n$ can't have a Hamiltonian base, it's longest cycle base is $\{C_n(i,\, j) | 1 \leq i < j \leq n\} \backslash \{C_n(n,\, 1)\}$ together with a (n-1)-cycle (1, 2, ...,n - 1).

Example 3 Let G be an outer planar triangular graph embedded in the sphere with its triangular faces $f_1, f_2, \ldots, f_{\varphi-1}$. Then it has exactly one Hamiltonian cycle ∂f_φ, here we use ∂f to denote the boundary of a face f. By Euler's formula, $\varphi - 1 = \beta(G)$, where φ is the number of faces. Let us define a set of cycles as following

$$C_n = \partial f_\varphi,$$

$$C_{n-1} = \partial f_1 \oplus \partial f_2 \oplus \cdots \oplus \partial f_{\varphi-2}, \quad C_{n-1}^{'} = \partial f_{\varphi-1} \oplus \partial f_{\varphi-2} \oplus \cdots \oplus \partial f_2$$

$$C_{n-2} = \partial f_1 \oplus \partial f_2 \oplus \cdots \oplus \partial f_{\varphi-3}, \quad C_{n-2}^{'} = \partial f_{\varphi-1} \oplus \partial f_{\varphi-2} \oplus \cdots \oplus \partial f_3$$

$$C_{n-k} = \partial f_1 \oplus \partial f_2 \oplus \cdots \oplus \partial f_{\varphi-k-1}$$

$$C_{n-k}^{'} = \partial f_{\varphi-1} \oplus \partial f_{\varphi-2} \oplus \cdots \oplus \partial f_{k+1}, \quad 1 \leq k \leq \varphi - 2.$$

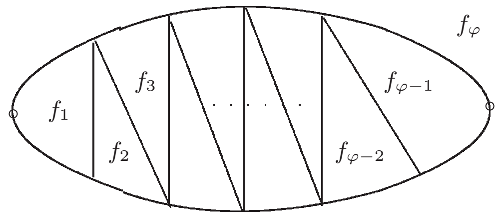

Fig. 2

$$\mathcal{B}=\begin{cases}\{C_n,C_{n-1},C_{n-2},\ldots,C_{\frac{n+3}{2}}\}\cup\{C'_{n-1},C'_{n-2},\ldots,C'_{\frac{n+3}{2}}\}, & \varphi\equiv 0\,(mod\ 2)\\ \\ \{C_n,C_{n-1},C_{n-2},\ldots,C_{\frac{n+4}{2}}\}\cup\{C'_{n-1},C'_{n-2},\ldots,C'_{\frac{n+2}{2}}\}, & \varphi\equiv 1\,(mod\ 2)\end{cases}$$

Thus $\mathcal{B}$ satisfies the condition of Theorem A. Hence, $\mathcal{B}$ is a longest cycle base, and the length of longest cycle base is

$$l(B)=\begin{cases}n+2(n-1)+2(n-2)+\cdots+2\left(\dfrac{n+3}{2}\right), & \varphi\equiv 0\,(mod\ 2)\\ \\ n+2(n-1)+2(n-2)+\cdots+2\left(\dfrac{n+4}{2}\right)+\dfrac{n+2}{2}, & \varphi\equiv 1\,(mod\ 2)\end{cases}$$

Example 4 Again we consider the " Möbius ladder graph " in Fig.1. It is clear that the edge–width(i.e., $ew(G)$) is $n+1$ and there are $n+1$ shortest non-separating cycles:

$$C_i=\begin{cases}(y_1,y_2,\ldots,y_i,x_i,x_{i+1},\ldots,x_n), & 1\le i\le n\\ \\ (y_1,y_2,\ldots,y_n,x_1), & i=n+1\end{cases}$$

Notice that $\beta(G)=n+1$ and $\{C_1,C_2,\ldots,C_{n+1}\}$ may generate every facial cycle and every non-contractible cycle of G. Thus, $\mathcal{B}=\{C_1,C_2,\ldots,C_n\}$ is a shortest non-separating cycle base with length $l(\mathcal{B})=(n+1)^2$. Although there are many such bases in G, they have the same structure as we have shown in Theorem J. Since our definition of non-separating cycles on locally orientable surface refuses the existence of facial cycles in such shortest non-separating cycle base, there may exist an edge contained in exactly one cycle in such a base. For instance, the edge (x_1,y_n) in Fig.1 is contained in exactly one non-separating cycle of such shortest cycle base.

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