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Surface Embeddability of Graphs via Joint Trees

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Abstract: This paper provides a way to observe embedings of a graph on surfaces based on join trees and then characterizations of orientable and nonorientable embeddabilities of a graph with given genus.

Key Words: Surface, graph, Smarandache λ^S-drawing, embedding, joint tree.

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§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache* λ^S*-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S.

The term *joint three* looks firstly appeared in [1] and then in [2] in a certain detail and [3] firstly in English. However, the theoretical idea was initiated in early articles of the author [4–5] in which maximum genus of a graph in both orientable and nonorientable cases were investigated.

The central idea is to transform a problem related to embeddings of a graph on surfaces i.e., compact 2-manifolds without boundary in topology into that on polyhegons (or polygons of even size with binary boundaries). The following two principles can be seen in [3].

Principle A *Joint trees of a graph have a 1–to–1 correspondence to embeddings of the graph with the same orientability and genus i.e., on the same surfaces.*

Principle B *Associate polyhegons (as surfaces) of a graph have a 1–to–1 correspondence to joint trees of the graph with the same orientability and genus, i.e., on the same surfaces.*

The two principle above are employed in this paper as the theoretical foundation. These enable us to discuss in any way among associate polyhegons, joint trees and embeddings of a graph considered.

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§2. Layers and Exchangers

Given a surface $S = (A)$. it is divided into segments layer by layer as in the following.

The 0th layer contains only one segment, *i.e.*, $A(= A_0)$;

The 1st layer is obtained by dividing the segment A_0 into l_1 segments, *i.e.*, $S = (A_1, A_2, \cdots, A_{l_1})$, where $A_1, A_2, \cdots, A_{l_1}$ are called the 1*st layer segments*;

Suppose that on $k-1$st layer, the $k-1$st layer segments are $A_{\underline{n}_{(k-1)}}$ where $\underline{n}_{(k-1)}$ is an integral $k-1$-vector satisfied by

$$\underline{1}_{(k-1)} \leqslant (n_1, n_2, \cdots, n_{k-1}) \leqslant \underline{N}_{(k-1)}$$

with $\underline{1}_{(k-1)} = (1, 1, \cdots, 1)$, $\underline{N}_{(k-1)} = (N_1, N_2, \cdots, N_{k-1})$, $N_1 = l_1 = N_{(1)}$, $N_2 = l_{A_{N_{(1)}}}$, $N_3 = l_{A_{\underline{N}_{(2)}}}$, $\cdots$, $N_{k-1} = l_{A_{\underline{N}_{(k-2)}}}$, then the kth layer segments are obtained by dividing each $k-1$st layer segment as

$$A_{\underline{n}_{(k-1)},1}, A_{\underline{n}_{(k-1)},2}, \cdots, A_{\underline{n}_{(k-1)},l_{A_{\underline{n}_{(k-1)}}}} \tag{1}$$

where $\underline{1}_{(k)} = (\underline{n}_{(k-1)}, 1) \leqslant (\underline{n}_{(k-1)}, i) \leqslant \underline{N}_{(k)} = (\underline{N}_{(k-1)}, N_k)$ and $N_k = l_{A_{\underline{N}_{(k-1)}}}$. Segments in (1) are called *successors* of $A_{\underline{n}_{(k-1)}}$. Conversely, $A_{\underline{n}_{(k-1)}}$ is the *predecessor* of any one in (1).

A layer segment which has only one element is called an *end segment* and others, *principle segments*. For an example, let

$$S = (1, -7, 2, -5, 3, -1, 4, -6, 5, -2, 6, 7, -3, -4).$$

Fig.2.1 shows a layer division of S and Tab.2.1, the principle segments in each layer.

For a layer division of a surface, if principle segments are dealt with vertices and edges are with the relationship between predecessor and successor, then what is obtained is a tree denoted by T. On T, by adding cotree edges as end segments, a graph $G = (V, E)$ is induced. For example, the graph induced from the layer division shown in Fig.1 is as

$$V = \{A, B, C, D, E, F, G, H, I\} \tag{2}$$

and

$$E = \{a, b, c, d, e, f, g, h, 1, 2, 3, 4, 5, 6, 7\}, \tag{3}$$

where

$$a = (A, B), b = (A, C), c = (A, D), d = (B, E),$$
$$e = (C, F), f = (C, G), g = (D, H), h = (D, I),$$

and

$$1 = (B, F), 2 = (E, H), 3 = (F, I), 4 = (G, I),$$
$$5 = (B, C), 6 = (G, H), 7 = (D, E).$$

By considering $E_T = \{a, b, c, d, e, f, g, h\}$, $\overline{E}_T = \{1, 2, 3, 4, 5, 6, 7\}$, $\delta_i = 0, i = 1, 2, \cdots, 7$, and the rotation σ implied in the layer division, a joint tree $\widehat{T}^{\delta}_{\sigma}$ is produced.

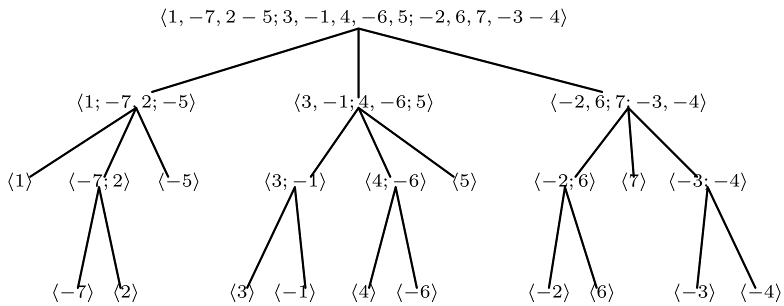

Fig.1 Layer division of surface S

Layers	Principle segments
0th layer	$A = \langle 1, -7, 2-5; 3, -1, 4, -6, 5; -2, 6, 7, -3-4\rangle$
1st layer	$B = \langle 1; -7, 2; -5\rangle, C = \langle 3, -1; 4, -6; 5\rangle,$
	$D = \langle -2, 6; 7; -3, -4\rangle$
2nd layer	$E = \langle -7; 2\rangle, F = \langle 3; -1\rangle, G = \langle 4; -6\rangle,$
	$H = \langle -2; 6\rangle, I = \langle -3; -4\rangle$

Tab.1 Layers and principle segments

Theorem 1 *A layer division of a polyhegon determines a joint tree. Conversely, a joint tree determines a layer division of its associate polyhegon.*

Proof For a layer division of a polyhegon as a polyhegon, all segments are treated as vertices and two vertices have an edge if, and only if, they are in successive layers with one as a subsegment of the other. This graph can be shown as a tree. Because of each non-end vertex with a rotation and end vertices pairwise with binary indices, this tree itself is a joint tree.

Conversely, for a joint tree, it is also seen as a layer division of the surface determined by the boundary polyhegon of the tree. □

Then, an operation on a layer division is discussed for transforming an associate polyhegon into another in order to visit all associate polyhegon without repetition.

A layer segment with all its successors is called a *branch* in the layer division. The operation of interchanging the positions of two layer segments with the same predecessor in a layer division is called an *exchanger*.

Lemma 1 *A layer division of an associate polyhegon of a graph under an exchanger is still a layer division of another associate polyhegon. Conversely, the later under the same exchanger becomes the former.*

Proof On the basis of Theorem 1, only necessary to see what happens by exchanger on a joint tree once. Because of only changing the rotation at a vertex for doing exchanger once, exchanger transforms a joint tree into another joint tree of the same graph. This is the first conclusion. Because of exchanger inversible, the second conclusion holds. □

On the basis of this lemma, an exchanger can be seen as an operation on the set of all associate surfaces of a graph.

Lemma 2 *The exchanger is closed in the set of all associate polyhegons of a graph.*

Proof From Theorem 1, the lemma is a direct conclusion of Lemma 1. □

Lemma 3 *Let $\mathcal{A}(G)$ be the set of all associate polyhegons of a graph G, then for any S_1, $S_2 \in \mathcal{A}(G)$, there exist a sequence of exchangers on the set such that S_1 can be transformed into S_2.*

Proof Because of exchanger corresponding to transposition of two elements in a rotation at a vertex, in virtue of permutation principle that any two rotation can be transformed from one into another by transpositions, from Theorem 1 and Lemma 1, the conclusion is done. □

If $\mathcal{A}(G)$ is dealt as the vertex set and an edge as an exchanger, then what is obtained in this way is called the *associate polyhegon graph* of G, and denoted by $\mathcal{H}(G)$. From Principle A, it is also called the *surface embedding graph* of G.

Theorem 2 *In $\mathcal{H}(G)$, there is a Hamilton path. Further, for any two vertices, $\mathcal{H}(G)$ has a Hamilton path with the two vertices as ends.*

Proof Since a rotation at each vertex is a cyclic permutation(or in short a cycle) on the set of semi-edges with the vertex, an exchanger of layer segments is corresponding to a transposition on the set at a vertex.

Since any two cycles at a vertex v can be transformed from one into another by $\rho(v)$ transpositions where $\rho(v)$ is the valency of v, *i.e.*, the order of cycle(rotation), This enables us to do exchangers from the 1st layer on according to the order from left to right at one vertex to the other. Because of the finiteness, an associate polyhegon can always transformed into another by $|\mathcal{A}(G)|$ exchangers. From Theorem 1 with Principles 1–2, the conclusion is done. □

First, starting from a surface in $\mathcal{A}(G)$, by doing exchangers at each principle segments in one layer to another, a Hamilton path can always be found in considering Theorem 2 and Theorem 1. Then, a Hamilton path can be found on $\mathcal{H}(G)$.

Further, for chosen $S_1, S_2 \in \mathcal{A}(G) = V(\mathcal{H}(G))$ adjective, starting from S_1, by doing exchangers avoid S_2 except the final step, on the basis of the strongly finite recursion principle, a Hamilton path between S_1 and S_2 can be obtained. In consequence, a Hamilton circuit can be found on $\mathcal{H}(G)$.

Corollary 1 *In $\mathcal{H}(G)$, there exists a Hamilton circuit.*

Theorem 2 tells us that the problem of determining the minimum, or maximum genus of graph G has an algorithm in time linear on $\mathcal{H}(G)$.

§3. Main Theorems

For a graph G, let $\mathcal{S}(G)$ be the the associate polehegons (or surfaces) of G, and $\mathbf{S}_p$ and $\mathbf{S}_{\widetilde{q}}$, the subsets of, respectively, orientable and nonorientable polyhegons of genus $p \geqslant 0$ and $q \geqslant 1$.

Then, we have

$$\mathcal{S}(G) = \sum_{p \geqslant 0} \mathbf{S}_p + \sum_{q \geqslant 1} \mathbf{S}_{\widetilde{q}}.$$

Theorem 3 *A graph G can be embedded on an orientable surface of genus p if, and only if, $\mathcal{S}(G)$ has a polyhegon in $\mathbf{S}_p$, $p \geqslant 0$. Moreover, for an embedding of G, there exist a sequence of exchangers by which the corresponding polyhegon of the embedding can be transformed into one in $\mathbf{S}_p$.*

Proof For an embedding of G on an orientable surface of genus p, from Theorem 1 there is an associate polyhegon in $\mathbf{S}_p$, $p \geqslant 0$. This is the necessity of the first statement.

Conversely, given an associate polyhegen in $\mathbf{S}_p$, $p \geqslant 0$, from Theorems 1–2 with Principles A and B, an embedding of G on an orientable surface of genus p can be done. This is the sufficiency of the first statement.

The last statement of the theorem is directly seen from the proof of Theorem 2. □

For an orientable embedding $\mu(G)$ of G, denote by $\widetilde{\mathbf{S}}_\mu$ the set of all nonorientable associate polyhegons induced from $\mu(G)$.

Theorem 4 *A graph G can be embedded on a nonorientable surface of genus $q(\geqslant 1)$ if, and only if, $\mathcal{S}(G)$ has a polyhegon in $\widetilde{\mathbf{S}}_q$, $q \geqslant 1$. Moreover, if G has an embedding $\widetilde{\mu}$ on a nonorientable surface of genus q, then it can always be done from an orientable embedding μ arbitrarily given to another orientable embedding μ' by a sequence of exchangers such that the associate polyhegon of $\widetilde{\mu}$ is in $\widetilde{\mathbf{S}}_{\mu'}$.*

Proof For an embedding of G on a nonorientable surface of genus q, Theorem 1 and Principle B lead to that its associate polyhegon is in $\mathbf{S}_q$, $q \geqslant 1$. This is the necessity of the first statement.

Conversely, let $S_{\widetilde{q}}$ be an associate polyhegon of G in $\widetilde{\mathbf{S}}_q$, $q \geqslant 1$. From Principles A and B, an embedding of G on a nonorietable surface of genus q can be found from $S_{\widetilde{q}}$. This is the sufficiency of the first statement.

Since a nonorientable embedding of G has exactly one under orientable embedding of G by Principle A, Theorem 2 directly leads to the second statement. □

§4. Research Notes

A. Theorems 1 and 2 enable us to establish a procedure for finding all embeddings of a graph G in linear space of the size of G and in linear time of size of $\mathcal{H}(G)$. The implementation of this procedure on computers can be seen in [6].

B. In Theorems 3 and 4, it is necessary to investigate a procedure to extract a sequence of transpositions considered for the corresponding purpose efficiently.

C. On the basis of the associate polyhegons, the recognition of operations from a polyhegon of genus p to that of genus $p+k$ for given $k \geqslant 0$ have not yet be investigated. However, for the case $k=0$ the operations are just Operetions 0–2 all topological that are shown in [1–3].

D. It looks worthful to investigate the associate polyhegon graph of a graph further for accessing the determination of the maximum(orientable) and minimum(orientable or nonorientable) genus of a graph.

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